# Weil-étale cohomology for $n < 0$

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**Conjecture**: meromorphic continuation to $s \in \mathbb{C}$.

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- **Dedekind**: $\zeta_F(s) = \zeta_{\operatorname{Spec}\mathcal{O}_F}(s)$ for a number field $F/\mathbb{Q}$.
- **Hasse–Weil**: $X/\mathbb{F}_q$, then

$$\zeta_X(s) = Z_X(q^{-s}),$$

where

$$Z_X(t) = \exp\left(\sum_{m\geq 1} \frac{\#X(\mathbb{F}_{q^m})}{m} t^m\right) \overset{\text{Dwork}}{\in} \mathbb{Q}(t).$$

(Cf. Weil conjectures.)

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- **Special value** (leading Taylor coefficient) at $s = n$:

$$\zeta_X^*(s) := \lim_{s \to n} (s-n)^{-d_n} \zeta_X(s).$$

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  $R_F :=$ **Dirichlet regulator** $\in \mathbb{R}$.
- Formulas for other $n \in \mathbb{Z}$? 🤔🤔🤔

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- Knudsen–Mumford determinants $\Longrightarrow$ canonical isomorphism

$$\lambda \colon \mathbb{R} \xrightarrow{\cong} \underbrace{(\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)))}_{\text{free } \mathbb{Z}\text{-mod of rk } 1} \otimes \mathbb{R}.$$

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- $\lambda(\zeta^*_X(n)^{-1}) \cdot \mathbb{Z} \stackrel{???}{=} \det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n))$.

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# From now on fix n < 0

# Motivic cohomology $H^\bullet(X_{ét}, \mathbb{Z}^c(n))$

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- **Conjecture** (Lichtenbaum): $H^i(X_{ét}, \mathbb{Z}^c(n))$ are finitely generated.

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$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \cong \left( \begin{array}{c} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \end{array} \right),$$

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  $\mathbb{R}(n) := (2\pi i)^n$, as a $G_\mathbb{R} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant sheaf.
- Long exact sequence of $H^i_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R}$: need a **regulator**.

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- **Kerr–Lewis–Müller-Stach** (2006) $\Longrightarrow$ for $X_{\mathbb{C}}$ is smooth and quasi-projective:

$$Reg\colon R\Gamma(X_{ét}, \mathbb{Z}^c(n)) \to R\Gamma_{BM}(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[1].$$

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$$Reg^{\vee}\colon R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \to R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{R})$$

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- Splitting over $\mathbb{R}$ + Beilinson's conjecture $\Longrightarrow$ l.e.s.

$$\cdots \to H^i_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \to H^{i+1}_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \to \cdots$$

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- **Then**

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- Note: this would imply
  $d_n = \sum_i (-1)^i \dim_{\mathbb{R}} H^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))$.

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- (Whenever makes sense) compatible with the **Tamagawa number conjecture** (Bloch−Kato−Fontaine−Perrin-Riou).
- Well-behaved under decompositions: for $Z \to X \leftarrow U$ holds $\zeta_X(s) = \zeta_Z(s) \cdot \zeta_U(s)$ (obviously), and in fact

$$\mathbf{C}(X, n) \iff \mathbf{C}(Z, n) + \mathbf{C}(U, n).$$

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Consider the étale sheaf $\mathbb{Z}(n) := \bigoplus_p \varinjlim_r j_{p!}\mu_{p^r}^{\otimes n}[-1]$, where $j_p \colon X[1/p] \hookrightarrow X$.

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 & & & & R\Gamma_{W,c}(X,\mathbb{Z}(n)) & & \\
 & & & & \downarrow & & \\
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]) & \longrightarrow & R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X,\mathbb{Z}(n)) & \longrightarrow & [+1] \\
\downarrow & & \downarrow{\scriptstyle \text{comparison}} & & \downarrow & & \downarrow \\
0 & \longrightarrow & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) & \overset{\text{id}}{\to} & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) & \longrightarrow & 0 \\
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- A less ad-hoc definition of Weil-étale complexes?
  Morally, there should be a Grothendieck topology behind everything.

# Thank you!