Weil-étale cohomology for n < 0

Alexey Beshenov
(CIMAT, Guanajuato)

November 29, 2019

First IMSA Conference
Centro de Colaboración Samuel Gitler / CINVESTAV

Arithmetic zeta-functions (Serre, 1965)

$$\begin{tikzcd} X \arrow[d, "\begin{array}{l}\text{separated,}\\ \text{finite type}\end{array}"] \\ \operatorname{Spec}\mathbb{Z} \end{tikzcd}$$

$$\zeta_X(s) := \prod_{\substack{x \in X \\ \text{closed}}} \frac{1}{1 - \#(\mathcal{O}_{X,x}/\mathfrak{m})^{-s}}. \quad (\operatorname{Re} s > \dim X)$$

Conjecture: meromorphic continuation to $s \in \mathbb{C}$.

Extensively studied cases

- **Riemann**: $\zeta(s) = \prod_p \frac{1}{1-p^{-s}} = \zeta_{\operatorname{Spec}\mathbb{Z}}(s)$.
- **Dedekind**: $\zeta_F(s) = \zeta_{\operatorname{Spec}\mathcal{O}_F}(s)$ for a number field $F/\mathbb{Q}$.
- **Hasse–Weil**: $X/\mathbb{F}_q$, then

  $$\zeta_X(s) = Z_X(q^{-s}),$$

  where

  $$Z_X(t) = \exp\left(\sum_{m\geq 1} \frac{\#X(\mathbb{F}_{q^m})}{m}\, t^m\right) \overset{\text{Dwork}}{\in} \mathbb{Q}(t).$$

  (Cf. Weil conjectures.)

Special values

- Fix $n \in \mathbb{Z}$.
- $d_n :=$ **vanishing order** of $\zeta_X(s)$ at $s = n$.
- **Special value** (leading Taylor coefficient) at $s = n$:

$$\zeta_X^*(n) := \lim_{s \to n} (s - n)^{-d_n} \zeta_X(s).$$

Classical motivation: class number formula

- Let $X = \operatorname{Spec} \mathcal{O}_F$ and $n = 0$.
- Zero of order $d_0 = r_1 + r_2 - 1$,
  where $r_1 := \#$ real places, $2r_2 := \#$ complex places.
- Special value $\zeta_F^*(0) = -\dfrac{\# H^1(\operatorname{Spec} \mathcal{O}_F, \mathbb{G}_m)}{\# H^0(\operatorname{Spec} \mathcal{O}_F, \mathbb{G}_m)_{tors}} R_F$,
  $R_F :=$ **Dirichlet regulator** $\in \mathbb{R}$.
- Formulas for other $n \in \mathbb{Z}$?

Weil-étale cohomology (Lichtenbaum, 2000s)

Conjectural cohomology theory.

- Groups $H^i_{W,c}(X, \mathbb{Z}(n)) = H^i(R\Gamma_{W,c}(X, \mathbb{Z}(n)))$.
- Perfectness: finitely generated and $= 0$ for $|i| \gg 0$.
- Long exact sequence

$$\cdots \to H^i_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \to H^{i+1}_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \to \cdots$$

- Knudsen–Mumford determinants $\Longrightarrow$ canonical isomorphism

$$\lambda \colon \mathbb{R} \xrightarrow{\cong} \underbrace{(\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)))}_{\text{free } \mathbb{Z}\text{-mod of rk } 1} \otimes \mathbb{R}.$$

- $d_n \overset{???}{=} \sum_i (-1)^i \cdot i \cdot \operatorname{rk}_{\mathbb{Z}} H^i_{W,c}(X, \mathbb{Z}(n))$.
- $\lambda(\zeta_X^*(n)^{-1}) \cdot \mathbb{Z} \overset{???}{=} \det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n))$.

Some work on Weil-étale cohomology

Lichtenbaum, 2005: + work by **Geisser**	$X/\mathbb{F}_q$ smooth	
Lichtenbaum, 2009:	$X = \operatorname{Spec} \mathcal{O}_F$	
Morin, 2014:	$X/\mathbb{Z}$ proper, regular,	$n = 0$
Flach, Morin, 2018:	$X/\mathbb{Z}$ proper, regular,	$n \in \mathbb{Z}$
–, 2018:	$X/\mathbb{Z}$ any...	$n < 0$

From now on fix $n < 0$

Motivic cohomology $H^\bullet(X_{ét}, \mathbb{Z}^c(n))$

- **Geisser**, 2010: **dualizing cycle complexes** $\mathbb{Z}^c(n)$. Complexes of abelian sheaves on $X_{ét}$.
- A variation of **Bloch's cycle complexes** (1986).
- Motivation: arithmetic duality theorems.
- Behaves as **Borel–Moore homology**: for $Z \to X \leftarrow U$

$$R\Gamma(Z_{ét}, \mathbb{Z}^c(n)) \to R\Gamma(X_{ét}, \mathbb{Z}^c(n)) \to R\Gamma(U_{ét}, \mathbb{Z}^c(n)) \to [+1]$$

- Calculations: few and hard...
- **Conjecture** (Lichtenbaum): $H^i(X_{ét}, \mathbb{Z}^c(n))$ are finitely generated.

Weil-étale complexes (after Flach and Morin)

- Assuming Lichtenbaum's conjecture, there exists a perfect complex $R\Gamma_{W,c}(X, \mathbb{Z}(n))$.
- Splitting over $\mathbb{R}$:

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \cong \left(\begin{array}{c} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \\ \oplus \\ R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n))[-1] \end{array} \right),$$

  $\mathbb{R}(n) := (2\pi i)^n \, \mathbb{R}$, as a $G_\mathbb{R} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant sheaf.
- Long exact sequence of $H^i_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R}$: need a **regulator**.

Regulator morphism

- **Kerr–Lewis–Müller-Stach** (2006) $\Longrightarrow$ for $X_{\mathbb{C}}$ is smooth and quasi-projective:

$$Reg\colon R\Gamma(X_{ét}, \mathbb{Z}^c(n)) \to R\Gamma_{BM}(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[1].$$

- Note: as always, $n < 0$, this is why the RHS is simple.
- **Conjecture** (Beilinson): the dual

$$Reg^\vee\colon R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \to R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{R})$$

is a quasi-isomorphism.

- Splitting over $\mathbb{R}$ + Beilinson's conjecture $\Longrightarrow$ l.e.s.

$$\cdots \to H^i_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \to H^{i+1}_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \to \cdots$$

Main conjecture $\mathbf{C}(X, n)$

- Assume...
  meromorphic continuation of $\zeta_X(s)$ around $s = n < 0$,
  $X_{\mathbb{C}}$ is smooth quasi-projective,
  Lichtenbaum's and Beilinson's conjectures.
- **Then**

$$d_n = \sum_i (-1)^i \cdot i \cdot \mathrm{rk}_{\mathbb{Z}} H^i_{W,c}(X, \mathbb{Z}(n)),$$

$$\lambda(\zeta_X^*(n)^{-1}) \cdot \mathbb{Z} = \det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)).$$

- Note: this would imply
  $d_n = \sum_i (-1)^i \dim_{\mathbb{R}} H^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))$.

What it's good for?

- If X is proper and regular, then $\mathbf{C}(X, n)$ is equivalent to the conjecture of Flach and Morin.
- (Whenever makes sense) compatible with the **Tamagawa number conjecture** (Bloch–Kato–Fontaine–Perrin-Riou).
- Well-behaved under decompositions: for $Z \to X \leftarrow U$ holds $\zeta_X(s) = \zeta_Z(s) \cdot \zeta_U(s)$ (obviously), and in fact

$$\mathbf{C}(X, n) \iff \mathbf{C}(Z, n) + \mathbf{C}(U, n).$$

* Construction (after Flach and Morin)

Consider the étale sheaf $\mathbb{Z}(n) := \bigoplus_p \varinjlim_r j_{p!}\mu_{p^r}^{\otimes n}[-1]$, where $j_p \colon X[1/p] \hookrightarrow X$.

$$\begin{tikzcd}
 & & R\Gamma_{W,c}(X,\mathbb{Z}(n)) \arrow[d] & \\
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]) \arrow[r] \arrow[d] & R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n)) \arrow[r] \arrow[d, "\text{comparison}"] & R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[r] \arrow[d, dashed] & {[+1]} \arrow[d] \\
0 \arrow[r] & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \arrow[r, "\text{id}"] & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \arrow[r] \arrow[d] & 0 \\
 & & R\Gamma_{W,c}(X,\mathbb{Z}(n))[1] &
\end{tikzcd}$$

Some questions

- A regulator for non-smooth $X_{\mathbb{C}}$?
- A less ad-hoc definition of Weil-étale complexes?
  Morally, there should be a Grothendieck topology behind everything.

Thank you!