Research project on Weil-étale cohomology

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In this note I will briefly describe what is Weil-étale cohomology, my contributions to its study, and plans for further research.

1 Zeta functions of schemes

Given a scheme X of finite type over $\operatorname{Spec}\mathbb{Z}$, one may attach to it the corresponding **zeta function**

$$\zeta_X(s) = \prod_{x\in|X|} \frac{1}{1-N(x)^{-s}}.$$

Here $|X|$ denotes the set of closed points of X, and for $x \in |X|$ the norm $N(x)$ is the order of the corresponding residue field $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$. The above product converges for $\operatorname{Re} s > \dim X$, and conjecturally, the zeta function admits a meromorphic continuation to the whole complex plane.

In particular, if $X = \operatorname{Spec}\mathcal{O}_F$ is the spectrum of the ring of integers of a number field $F/\mathbb{Q}$, then $\zeta_X(s) = \zeta_F(s)$ is the Dedekind zeta function studied extensively in algebraic number theory. If X is a smooth projective variety over a finite field $\mathbb{F}_q$, then $\zeta_X(s) = Z_X(q^{-s})$, where

$$Z_X(t) = \exp\Big(\sum_{n\ge1} \frac{\#X(\mathbb{F}_{q^n})}{n}\, t^n\Big)$$

is the Hasse–Weil zeta function, whose basic properties are given by Weil conjectures in algebraic geometry (see e.g. [Kat1994]).

For basic facts and conjectures about zeta functions of schemes, I refer to Serre's survey [Ser1965].

Of particular interest are the so-called **special values** of $\zeta_X(s)$ at integers $s = n \in \mathbb{Z}$. Namely, if d_n is the vanishing order of $\zeta_X(s)$ at $s = n$, then the corresponding special value is defined to be the leading Taylor coefficient at $s = n$:

$$\zeta_X^*(n) = \lim_{s\to n}(s-n)^{d_n}\,\zeta_X(s)$$

(assuming the analytic continuation around $s = n$).

2 Weil-étale cohomology

Both vanishing order d_n and special value $\zeta_X^*(n)$ are expected to have expressions in terms of *certain invariants* attached to X. There are various conjectures, of varying generality, that make this precise. I am interested in a relatively recent **Weil-étale cohomology** program, initiated by Stephen Lichtenbaum [Lic2005, Lic2009a, Lic2009b]. Other results for the case of varieties over finite fields have been obtained by Geisser [Gei2004, Gei2006].

Let me briefly explain what one expects from Weil-étale cohomology. Let X be a separated scheme of finite type over $\operatorname{Spec}\mathbb{Z}$. Then for a fixed integer n, Weil-étale cohomology consists of abelian groups $H^i_{W,c}(X,\mathbb{Z}(n))$ with the following conjectural properties.

W1) $H^i_{W,c}(X,\mathbb{Z}(n))$ are finitely generated abelian groups, trivial for $|i| \gg 0$.

As a consequence, to these groups one can associate the corresponding **determinant** $\det_\mathbb{Z} H^\bullet_{W,c}(X,\mathbb{Z}(n))$ in the sense of [KM1976], which is a free $\mathbb{Z}$-module of rank 1.

W2) After tensoring these cohomology groups with $\mathbb{R}$, one obtains a long exact sequence of finite-dimensional real vector spaces

$$\cdots \to H^{i-1}_{W,c}(X,\mathbb{Z}(n)) \otimes \mathbb{R} \xrightarrow{\smile\theta} H^{i}_{W,c}(X,\mathbb{Z}(n)) \otimes \mathbb{R} \xrightarrow{\smile\theta} H^{i+1}_{W,c}(X,\mathbb{Z}(n)) \otimes \mathbb{R} \to \cdots$$

By well-known properties of determinants of complexes, this induces a *canonical* isomorphism

$$\lambda\colon \mathbb{R} \xrightarrow{\cong} \left(\det_{\mathbb{Z}} H^{\bullet}_{W,c}(X,\mathbb{Z}(n))\right) \otimes \mathbb{R}.$$

W3) The vanishing order of $\zeta_X(s)$ at $s = n \in \mathbb{Z}$ is conjecturally given by

$$d_n = \sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot \operatorname{rk}_{\mathbb{Z}} H^i_{W,c}(X,\mathbb{Z}(n)).$$

W4) The corresponding special value is determined up to sign by

$$\lambda(\zeta^*_X(n)^{-1}) \cdot \mathbb{Z} = \det_{\mathbb{Z}} H^{\bullet}_{W,c}(X,\mathbb{Z}(n)).$$

Baptiste Morin gave in [Mor2014] a construction of Weil-étale cohomology for X a separated scheme of finite type, proper and regular, and $n = 0$. Later on this construction was generalized together with Matthias Flach in [FM2018] to any $n \in \mathbb{Z}$, under the same assumptions on X.

3 My work on Weil-étale cohomology

In my PhD thesis [Bes2018], co-supervised by Baptiste Morin and Bas Edixhoven, I generalized the work of Flach and Morin to any X that is separated and of finite type over $\operatorname{Spec}\mathbb{Z}$ (thus removing the assumption that X is proper or regular), while considering the case of $n < 0$ (which turns out to simplify certain aspects of the theory).

The main building block of $H^i_{W,c}(X,\mathbb{Z}(n))$ is the (étale) **motivic cohomology** $H^i_{ét}(X,\mathbb{Z}^c(n))$, defined in terms of "dualizing cycle complexes" $\mathbb{Z}^c(n)$, as introduced by Geisser in [Gei2010], and an **arithmetic duality theorem** in terms of $\mathbb{Z}^c(n)$, also due to Geisser (ibid.).

The precise construction of $H^i_{W,c}(X,\mathbb{Z}(n))$ is quite technical, so I refer to my preprint [Bes2020] for further details, and to the work of Flach and Morin [FM2018] for the case of X proper and regular. What is important is that

- the property W1) above is established assuming finite generation of motivic cohomology $H^i_{ét}(X,\mathbb{Z}^c(n))$;
- the sequence of real vector spaces in W2) is defined in terms of the **regulator** map; the exactness in W2) follows from the standard conjecture about the regulator.

As most of the formulas for special values, all this is hugely conjectural, especially at the level of generality we are interested in. A compelling evidence in favor of the above special value conjecture is that, whenever the comparison makes sense, it is equivalent to the **Tamagawa number conjecture** (**TNC**) of Bloch–Kato–Fontaine–Perrin-Riou [FPR1994].

One new interesting point is the following. If $Z \subset X$ is a closed subscheme and $U = X \setminus Z$ is its open complement, then $\zeta_X(s) = \zeta_Z(s)\zeta_U(s)$. Accordingly, one should expect the special value conjecture to be compatible with such "open-closed decompositions" of schemes: the special value conjecture for X should be equivalent to the corresponding conjecture for Z and U. I prove in my thesis that this is indeed the case in my situation: morally, this comes from a long exact sequence

$$\cdots \to H^i_{W,c}(U,\mathbb{Z}(n)) \to H^i_{W,c}(X,\mathbb{Z}(n)) \to H^i_{W,c}(Z,\mathbb{Z}(n)) \to H^{i+1}_{W,c}(U,\mathbb{Z}(n)) \to \cdots$$

Similarly, it is not difficult to see that one has $\zeta_{\mathbb{A}^r_X}(s) = \zeta_X(s-r)$, and thus a special value conjecture for the affine bundle $\mathbb{A}^r_X$ at $s = n$ should be equivalent to the corresponding conjecture for X at $s = n - r$. I prove that this is also the case.

As a result, one can start from certain very special cases of X for which the special value conjecture is known (e.g. using the equivalence with TNC and known cases of the latter), and then build new schemes (not necessarily proper or regular) for which the conjecture holds as well, unconditionally.

4 My research projects

Here I will list various problems that I have in mind for my future work.

1. It is interesting to write down specific formulas for particular cases of X that follow from the Weil-étale formalism.

   For instance, Jordan and Poonen write down in [JP2020] a formula for $\zeta_X^*(1)$, where X is any affine one-dimensional scheme of finite type over $\operatorname{Spec}\mathbb{Z}$. Using my machinery, similar formulas could be produced for $\zeta_X^*(n)$ and $n < 0$. (E.g. TNC is known for $X = \operatorname{Spec}\mathcal{O}_F$, where $F/\mathbb{Q}$ is an abelian number field.)

   This is the project I am currently working on.

2. The actual construction of Weil-étale cohomology is not formulated in terms of separate groups $H^i_{W,c}(X,\mathbb{Z}(n))$, but in terms of complexes $R\Gamma_{W,c}(X,\mathbb{Z}(n))$. At present, these are defined (both in [FM2018] and [Bes2020]) up to a *non-unique* isomorphism in the derived category $\mathbf{D}(\mathbb{Z})$, as a mapping fiber of certain canonical morphism in $\mathbf{D}(\mathbb{Z})$.

   This is not a big issue for the special value conjecture, since the determinants $\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)) = \det_{\mathbb{Z}} H^\bullet_{W,c}(X,\mathbb{Z}(n))$ are uniquely defined, but nevertheless, it would be useful to find a more canonical definition for $R\Gamma_{W,c}(X,\mathbb{Z}(n))$.

   This could be probably remedied using dg-categories [Toë11] or Lurie's stable ∞-categories [Lur2009].

3. In the above exposition, I swept under the rug some details about regulators. I use the construction of Kerr, Lewis, and Müller-Stach from [KLMS2006]. In general, the regulator has to do with complex points $X(\mathbb{C})$, and the regulator in [KLMS2006] is defined for $X_{\mathbb{C}}$ being smooth and quasi-projective. This is quite unfortunate, since my construction of Weil-étale cohomology $H^i_{W,c}(X,\mathbb{Z}(n))$ works for any X that is separated and of finite type over $\operatorname{Spec}\mathbb{Z}$ (assuming finite generation of the corresponding motivic cohomology). It would be interesting to find appropriate generalizations of the regulator for singular $X_{\mathbb{C}}$, and connect these to my machinery.

   There are variants of motivic cohomology for singular complex varieties, defined in terms of hyperresolutions, e.g. Hanamura's "Chow cohomology groups" [Han2000], that allow to formally extend the formula of Kerr–Lewis–Müller-Stach to the singular case. However, it seems like the singular regulators haven't been considered thoroughly for special value conjectures.

4. The Tamagawa number conjecture has a generalization, known as the **equivariant Tamagawa number conjecture** (**ETNC**); see e.g. Flach's survey [Fla2004]. Similarly, Weil-étale cohomology should also have an "equivariant refinement", and it would be interesting to write it down for my construction, and prove its compatibility with ETNC.

These are a few specific questions that naturally arise from my previous work. In general, Weil-étale cohomology is an active research topic, and there are still many open problems related to it.

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