$\boxed{07/05}$ Teorema de Dirichlet: $mcd(a,m)=1 \Rightarrow \exists$ número ∞ de primos p t.q $p \equiv a \ (m)$.
$(1/\varphi(m))$

§ Series de Dirichlet

def. $f: \mathbb{Z}_{>0} \to \mathbb{C} \rightsquigarrow F(s) = \sum_{n\geq 1} \frac{f(n)}{n^s}$.

$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$

1) abscisa de convergencia σ_0 t.q. $F(s)$ converge para $Re\, s > \sigma_0$

2) abscisa de conv. absoluta σ_a $\quad \sigma_0 \leq \sigma_a \leq \sigma_0 + 1$.

3) Si $\exists\, C$ t.q. $\forall N$ $\left|\sum_{1\leq n\leq N} f(n)\right| \leq C \Rightarrow F(s)$ converge para $Re\, s > 0$.

4) Si f es una función multiplicativa, $(f(ab)=f(a)\cdot f(b))$

$$F(s) = \prod_p \frac{1}{1-f(p)p^{-s}} \quad \text{cuando tenemos conv. absoluta}$$

$$\prod_p \left(\sum_{e\geq 0} \frac{f(p^e)}{p^{es}}\right) = \sum_{n\geq 1} \frac{f(n)}{n^s}$$

def Un carácter de Dirichlet mód m. $\quad \chi: (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$
$\rightsquigarrow \tilde{\chi}: \mathbb{Z} \to \mathbb{C}$.

$$\tilde{\chi}(a) = \begin{cases} \chi(a), & \text{si } mcd(a,m)=1 \\ 0, & \text{si } mcd(a,m)\neq 1. \end{cases}$$

$Hom((\mathbb{Z}/m\mathbb{Z})^\times, \mathbb{C}^\times)$ - gpo abeliano finito $\varphi(m)$

En gral, si G es un gpo abeliano finito,
$\hat{G} = Hom(G, \mathbb{C}^\times)$. $\quad G \simeq \hat{G}$

$$\sum_{\chi\in\hat{G}} \chi(g)\,\chi^{-1}(h) = \begin{cases} |G|, & \text{si } g=h \\ 0, & \text{si } g\neq h \end{cases}$$ relación de ortogonalidad.

Si χ es un carácter de Dichlet mód m: $\quad mcd(a,m)=mcd(b,m)=1$

$$\sum_\chi \chi(a)\cdot\chi^{-1}(b) = \begin{cases} \varphi(m), & \text{si } a\equiv b \ (m) \\ 0, & \text{si } a\not\equiv b \ (m) \end{cases}$$

def. Si χ es un carácter de Dirichlet, la serie L

$$L(s,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s}. \qquad \chi: (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$$

$\zeta(s)$ converge para $\operatorname{Re} s > 1$.

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}} \qquad (\operatorname{Re} s > 1).$$

$L(s,\chi)$ converge absolutamente para $\operatorname{Re} s > 1$

$$L(s,\chi) = \prod_p \frac{1}{1-\chi(p)\cdot p^{-s}} \qquad (\operatorname{Re} s > 1)$$

$$\zeta(1) = \sum_{n\geq 1} \frac{1}{n} = \infty$$

$$\lim_{s\to 1^+} (s-1)\zeta(s) = 1.$$

·) Si $\chi = \chi_0$ es el carácter trivial $\chi_0(n) = \begin{cases} 1, & (n,m)=1 \\ 0, & (n,m)\neq 1 \end{cases}$

$$L(s,\chi_0) = \prod_{p\nmid m} \frac{1}{1-p^{-s}} = \prod_{p\mid m}(1-p^{-s})\cdot \underbrace{\zeta(s)}$$

$$\lim_{s\to 1^+} (s-1)L(s,\chi_0) = \prod_{p\mid m}\left(1-\frac{1}{p}\right) = \frac{\varphi(m)}{m}$$

·) Si $\chi \neq \chi_0$, entonces $\displaystyle\sum_{a\in(\mathbb{Z}/m\mathbb{Z})^\times} \chi(a) = 0.$

$$\chi(b)\neq 1 \Rightarrow \sum_a \chi(ab) = \chi(b)\cdot\sum_a \chi(a)$$

$$\Vert$$

$$\sum_a \chi(a)$$

$$\forall N \quad \left|\sum_{1\leq n\leq N} \chi(n)\right| < m. \Rightarrow L(s,\chi) \text{ converge para } \operatorname{Re} s > 0.$$

Teorema Si $\chi \neq \chi_0$, entonces $L(1,\chi)\neq 0$.

Ejemplo $m=3$, $\chi: (\mathbb{Z}/3\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$

$$1 \longmapsto +1$$
$$2 \longmapsto -1$$

$$L(1,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n} = \sum_{n\geq 0}\left(\frac{1}{(3n+1)} - \frac{1}{(3n+2)}\right) = \sum_{n\geq 0} \frac{1}{(3n+1)(3n+2)}$$

$$\frac{1}{(3n+1)(3n+2)} = \int_0^1 t^{3n}(1-t)\,dt.$$

$$L(1,\chi) = \int_0^1 \left(\sum_{n\geq 0} t^{3n}(1-t)\right) dt = \int_0^1 \frac{1-t}{1-t^3}\,dt$$

$$= \int_0^1 \frac{1}{1+t+t^2}\,dt = \frac{2}{\sqrt{3}}\left[\arctan\left(\frac{2t+1}{\sqrt{3}}\right)\right]_0^1$$

$$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\pi}{3\sqrt{3}}$$

§ Densidad de primos

Lema 1 $\lim_{s\to 1^+} \log\zeta(s) \Big/ \log\frac{1}{s-1} = 1.$ (*)

Dem $\psi(s) := (s-1)\zeta(s)$. $\log\psi(s) = \log(s-1) + \log\zeta(s)$

$$1 + \log\psi(s)\Big/\log\frac{1}{s-1} = \log\zeta(s)\Big/\log\frac{1}{s-1}$$

$$\lim_{s\to 1^+}\psi(s) = 1 \implies (*)$$

$\square$

Lema 2 $\log\zeta(s) = \sum_p \frac{1}{p^s} + R(s)$,

donde $R(s)$ esté acotado, cuando $s\to 1^+$.

Dem $\zeta(s) = \prod_{p\leq N}\frac{1}{1-p^{-s}}\cdot R_N(s)$, $\quad R_N(s)\xrightarrow[N\to\infty]{} 1$.

$$\log\zeta(s) = \sum_{p\leq N} -\log(1-p^{-s}) + \log R_N(s).$$

$$\boxed{\sum_{n\geq 1}\frac{x^n}{n} = -\log(1-x) \quad -1<x<1}$$

$$= \sum_{p\leq N}\sum_{n\geq 1}\frac{p^{-ns}}{n} + \log R_N(s).$$

$$= \sum_{p\leq N}\frac{1}{p^s} + \sum_{p\leq N}\sum_{n\geq 2}\frac{p^{-ns}}{n} + \log R_N(s)$$

$N\to\infty$: $\log\zeta(s) = \sum_p \frac{1}{p^s} + \underbrace{\sum_p\sum_{n\geq 2}\frac{p^{-ns}}{n}}_{=R(s)}$

$$R(s) = \sum_p' p^{-2s} \cdot \underbrace{\frac{1}{1-p^{-s}}}_{\leq 2 \ \forall p \ \forall s > 1} \leq 2 \cdot \sum_p \frac{1}{p^{2s}} < 2 \cdot \underline{\zeta(2s)}.$$

$\square$

$$\underline{\log \zeta(s)} = \sum_p \frac{1}{p^s} + \underbrace{R(s)}_{\text{acotado.}}$$

$$\sum_p \frac{1}{p} = \infty.$$

<u>**def**</u> Sea X un conjunto que consiste en números primos. La <u>densidad</u> de X viene dada por

$$d(X) = \lim_{s \to 1^+} \sum_{p \in X} \frac{1}{p^s} \Big/ \underbrace{\log \frac{1}{s-1}}$$

$\log \zeta(s)$

1) X es un conjunto finito, $d(X) = 0$.
2) X consiste en todos los primos, salvo un # finito, $d(X) = 1$.
3) Si $X = Y \sqcup Z$ unión disjunta $\Rightarrow$ $d(Y)$ y $d(Z)$ existen $\Rightarrow d(X) = d(Y) + d(Z)$.

<u>**Teorema de Dirichlet**</u> Si $X = \{p \text{ primo} \mid p \equiv a \ (m)\}$, $(a, m) = 1$, entonces $d(X) = 1/\varphi(m)$.

$$\begin{cases} p \equiv 1 \ (3) \rightsquigarrow 1/2 \\ p \equiv 2 \ (3) \rightsquigarrow 1/2. \end{cases}$$

{ <u>**Bosquejo de la prueba**</u>

$$\chi \neq \chi_0 \Rightarrow L(1, \chi) \neq 0.$$

Si χ un carácter mód m,

$$G(s, \chi) = \sum_p' \sum_{k \geq 1}' \frac{(\chi(p) \cdot p^{-s})^k}{k}$$ converge para $s > 1$.

$$\exp(G(s, \chi)) = L(s, \chi).$$

$$\exp\left(\sum_{k \geq 1}' \frac{(\chi(p)\, p^{-s})^k}{k}\right) = \exp\left(-\log(1 - \chi(p)\, p^{-s})\right) = \frac{1}{1 - \chi(p)\, p^{-s}}$$

Proposición $\lim_{s\to 1^+} G(s,\chi) \Big/ \log \frac{1}{s-1} = \begin{cases} 1, & \text{si } \chi = \chi_0 \\ 0, & \text{si } \chi \neq \chi_0. \end{cases}$

Dem $G(s,\chi_0) = \log L(s,\chi_0) = \log\left(\prod_{p\mid m} (1-p^{-s}) \cdot \zeta(s)\right)$

·) si $\chi = \chi_0$

$$= \underbrace{\sum_{p\mid m} \log(1-p^{-s})}_{} + \log \zeta(s).$$

·) Si $\chi \neq \chi_0$. $G(1,\chi) = \log \underbrace{L(1,\chi)}_{\neq 0.}$ (Salvo $2\pi i\mathbb{Z}$)

De todas formas, $G(1,\chi)$ es acotado. ☐

$$G(s,\chi) = \text{“}\log L(s,\chi)\text{”}$$

$$= \sum_{p\nmid m} \frac{\chi(p)}{p^s} + \underbrace{R_\chi(s)}_{}.$$

↑ acotado para $s \to 1^+$

Si $\text{mcd}(a,m)=1$, multiplicamos por $\chi^{-1}(a)$ y sumamos dos pe χ todos los caracteres de Dirichlet mód m.

$$\sum_\chi \bar{\chi}^1(a)\, G(s,\chi) = \sum_{p\nmid m} \frac{1}{p^s} \cdot \underbrace{\sum_\chi \chi(p)\chi^{-1}(a)}_{} + \underbrace{\sum_\chi \chi^{-1}(a) R_\chi(s)}_{}$$

$$\longrightarrow = \begin{cases} \varphi(m), & \text{si } p \equiv a\ (m) \\ 0, & \text{si } p \not\equiv a\ (m) \end{cases}$$

$$\sum_\chi \bar{\chi}^1(a)\, \underline{G(s,\chi)} = \varphi(m) \sum_{p\equiv a(m)}{}' \frac{1}{p^s} + \left(\underline{\cdots}\right)$$

Dividir por $\log \frac{1}{s-1}$ y tomar $s \to 1^+$

$$1 = \varphi(m) \cdot \underbrace{\lim_{s\to 1^+} \sum_{p\equiv a(m)} \frac{1}{p^s} \Big/ \log \frac{1}{s-1}}_{d(\chi).}$$

Conclusión: $d(X) = 1/\varphi(m)$.

§ **Densidad natural** X - conjunto de primos.

$$d_{nat}(X) = \lim_{N\to\infty} \frac{\#\{p \in X \mid p \leq N\}}{\#\{p \text{ primo} \mid p \leq N\}}.$$

Si $d_{nat}(X)$ existe, entonces $d_{nat}(X) = d(X)$.

$\not\Leftarrow$

$$\pi(a,m,N) = \#\{p \mid p \leq N,\ p \equiv a \ (m)\}.$$

Teorema: $\pi(a,m,N) \sim \frac{1}{\varphi(m)} \cdot \frac{N}{\log N}$

Teorema de los números primos:

$$\pi(N) = \#\{p \text{ primo} \mid p \leq N\}$$

$$\pi(N) \sim \frac{N}{\log N}.$$

T. de Dichlet: 1837.

T. de los # primos: 1896.