$\boxed{21/10}$

$$\Lambda \subset \underset{\underset{X}{\cup}}{V}/\mathbb{R} \qquad \text{vol}\, X > 2^n \cdot \text{covol.}\, \Lambda \implies \omega \in \Lambda,\ \omega \neq 0,\ \omega \in X$$

$K/\mathbb{Q}$ campo de números $\quad n = [K:\mathbb{Q}]$, $\quad \tau : K \hookrightarrow \mathbb{C}$

$\rho : K \hookrightarrow \mathbb{R}$ (r_1) $\qquad \sigma : K \hookrightarrow \mathbb{C}$, $\bar\sigma \neq \sigma$ ($2r_2$)

$r_1 + 2r_2 = n$.

$$K_{\mathbb{C}} = \prod_{\tau} \mathbb{C} \qquad \Phi : K \hookrightarrow K_{\mathbb{C}},\quad \alpha \longmapsto (\tau(\alpha))_\tau$$

$$\langle z, z' \rangle = \sum_{\tau}{}' z_\tau \cdot \overline{z'_\tau}$$ producto hermitiano.

$$\langle z', z \rangle = \overline{\langle z, z' \rangle} \qquad \langle z, z \rangle > 0 \ \forall z \neq 0.$$

$$G_{\mathbb{R}} = \text{Gal}(\mathbb{C}/\mathbb{R}) \curvearrowright K_{\mathbb{C}} \qquad F : K_{\mathbb{C}} \to K_{\mathbb{C}}$$ automorfismo $\mathbb{R}$-lineal

$$(z_\tau)_\tau \mapsto (\bar z_{\bar\tau})_\tau$$

$$K_{\mathbb{R}} = K_{\mathbb{C}}^{G_{\mathbb{R}}} = \{ (z_\tau)_\tau \mid z_{\bar\tau} = \bar z_\tau \}$$

$$\langle Fz, Fz' \rangle = \overline{\langle z, z' \rangle}$$

$\langle \cdot, \cdot \rangle$ se restringe a un producto escalar sobre $K_{\mathbb{R}}$

$x, y \in K_{\mathbb{R}}$.

$$\begin{cases} \langle x, y \rangle = \langle Fx, Fy \rangle = \overline{\langle x, y \rangle} \\ \langle y, x \rangle = \overline{\langle x, y \rangle} = \langle x, y \rangle \\ \langle x, x \rangle > 0 \quad \text{si } x \neq 0 \end{cases}$$

$$\bar\tau(\alpha) = \overline{\tau(\alpha)}$$

$$\begin{tikzcd} K \arrow[r, hook, "\Phi"] \arrow[dr, "\Phi"'] & K_{\mathbb{R}} \arrow[d, hook] \\ & K_{\mathbb{C}} \end{tikzcd} \qquad \alpha \mapsto (\tau(\alpha))_\alpha$$

Proposición $\underset{\Lambda}{\Phi(\mathcal{O}_K)} \subseteq K_{\mathbb{R}}$ es un retículo de rango completo t.q. covol. $\Lambda = \sqrt{|\Delta_K|}$

Demostración $\mathcal{O}_K = \mathbb{Z}\alpha_1 + \dots + \mathbb{Z}\alpha_n$.

$$\Lambda = \mathbb{Z}\Phi(\alpha_1) + \dots + \mathbb{Z}\Phi(\alpha_n)$$

$$\tau_i : K \hookrightarrow \mathbb{C} \qquad \Delta_K = \det(A)^2 \qquad A = (\tau_i \alpha_j)_{i,j}$$

$$\left(\langle \varphi(\alpha_i), \varphi(\alpha_j)\rangle\right)_{i,j} = \left(\sum_k{}' \tau_k(\alpha_i)\cdot \overline{\tau_k(\alpha_j)}\right)_{i,j}$$

$$= A\cdot \overline{A}^t$$

$$\operatorname{covol}\Lambda = \sqrt{\left|\det(A\cdot\overline{A}^t)\right|} = \sqrt{|\Delta_K|} \qquad \boxtimes$$

Corolario $I \subseteq \mathcal{O}_K$ ideal no nulo.

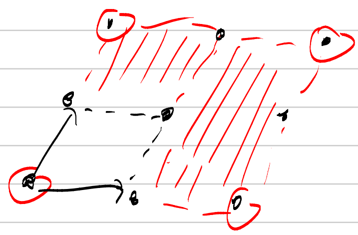
$$\Lambda = \varphi(I) \subseteq K_{\mathbb{R}}. \qquad \operatorname{covol}\Lambda = \sqrt{|\Delta_K|}\cdot N_{K/\mathbb{Q}}(I)$$

Dem Si $I \neq 0$ $\qquad N_{K/\mathbb{Q}}(I) \overset{\text{def}}{=} [\mathcal{O}_K : I] = \#(\mathcal{O}_K/I)$

$$\begin{array}{ccc} \varphi(I) & \subseteq & \varphi(\mathcal{O}_K) \\ \| & & \| \\ \Lambda & \subseteq & \Lambda' \end{array} \qquad [\Lambda' : \Lambda] = [\mathcal{O}_K : I] = N_{K/\mathbb{Q}}(I).$$

$$\boxed{\operatorname{covol}(\Lambda) = \operatorname{covol}(\Lambda')\cdot[\Lambda':\Lambda]}$$

$\boxtimes$



Ejemplo $\mathcal{O}_K = \mathbb{Z}[\zeta_3]$, $\quad K = \mathbb{Q}(\zeta_3)$ $\underset{\overline{\sigma}}{\overset{\sigma}{\rightrightarrows}} \mathbb{C}$ $\qquad \sigma: \zeta_3 \mapsto \zeta_3, \quad \overline{\sigma}: \zeta_3 \mapsto \zeta_3^2$

$$K_{\mathbb{R}} = \{(z_\sigma, z_{\overline{\sigma}}) \in K_{\mathbb{C}} \mid z_{\overline{\sigma}} = \overline{z_\sigma}\}$$

$$K_{\mathbb{R}} \simeq \mathbb{R}^2$$

$$(z_\sigma, z_{\overline{\sigma}}) \longmapsto (x_\sigma, x_{\overline{\sigma}}) = (\operatorname{Re} z_\sigma, \operatorname{Im} z_\sigma)$$

$$z_\sigma = x_\sigma + i y_\sigma, \qquad z'_\sigma = x'_\sigma + i y'_\sigma$$

$$z_\sigma \overline{z'_\sigma} + z_{\overline{\sigma}}\cdot\overline{z'_{\overline{\sigma}}} = z_\sigma \overline{z'_\sigma} + \overline{z_\sigma}\cdot z'_\sigma$$

$$= 2(x_\sigma x'_\sigma + y_\sigma y'_\sigma).$$

$$= 2(x_\sigma x'_\sigma + x_{\overline{\sigma}}\cdot x'_{\overline{\sigma}})$$

$$\langle x, x'\rangle = \underline{2}\cdot(x_\sigma\cdot x'_\sigma + x_{\overline{\sigma}}\cdot x'_{\overline{\sigma}})$$

!!!

$$\varphi : \mathbb{Z}[\zeta_3] \hookrightarrow K_{\mathbb{R}} \simeq \mathbb{R}^2$$
$$1 \longmapsto (1,0)$$
$$\zeta_3 \longmapsto (\mathrm{Re}\,\zeta_3, \mathrm{Im}\,\zeta_3).$$

$$\left| \det \begin{pmatrix} \langle \varphi(1), \varphi(1)\rangle & \langle \varphi(1), \varphi(\zeta_3)\rangle \\ \langle \varphi(\zeta_3), \varphi(1)\rangle & \langle \varphi(\zeta_3), \varphi(\zeta_3)\rangle \end{pmatrix} \right|^{1/2}$$

$$= \left| \det \begin{pmatrix} 2 & 2\cdot \mathrm{Re}\,\zeta_3 \\ 2\cdot \mathrm{Re}\,\zeta_3 & 2\cdot|\zeta_3|^2 \end{pmatrix} \right|^{1/2} = \sqrt{4 - 4\cdot \mathrm{Re}(\zeta_3)^2} = \sqrt{3}$$

$$\Delta_{\mathbb{Q}(\zeta_3)} = \Delta_{\mathbb{Q}(\sqrt{-3})} = \underline{-3} \qquad \boxtimes$$

§ Cota de Minkowski

ρ encajes reales

σ encajes complejos

$$K_{\mathbb{R}} = \{ (z_\tau)_\tau \in K_{\mathbb{C}} \mid z_\rho \in \mathbb{R},\ z_{\bar\sigma} = \bar z_\sigma \}$$

$$(z_\tau) \qquad K_{\mathbb{R}} \simeq \mathbb{R}^n = \mathbb{R}^{r_1+2r_2}$$

$$(x_\tau) \longleftrightarrow \begin{cases} x_\rho = z_\rho \\ x_\sigma = \mathrm{Re}(z_\sigma) \\ x_{\bar\sigma} = \mathrm{Im}(z_\sigma) \end{cases}$$

$$\langle x, y\rangle = \sum_\tau \underline{\alpha_\tau}\, x_\tau y_\tau, \qquad \alpha_\tau = \begin{cases} 1, & \text{si } \tau \text{ es real} \\ 2, & \text{si } \tau \text{ es complejo} \end{cases}$$

$$K_{\mathbb{R}} \simeq \underline{\mathbb{R}^{r_1+2r_2}}$$
$$X \simeq \varphi(X), \quad X \subseteq K_{\mathbb{R}},\ \varphi(X) \subseteq \mathbb{R}^{r_1+2r_2}$$

$$\boxed{\mathrm{vol}(X) = 2^{r_2}\cdot \mathrm{vol}_{\mathrm{Leb}}(\varphi(X))}$$

$\underline{t>0}$ $$X_t = \{ (z_\tau) \in K_{\mathbb{R}} \mid \sum_\tau |z_\tau| \le t \}.$$

conjunto convexo simétrico

Lema $\operatorname{vol}(X_t) = 2^{r_1} \cdot \pi^{r_2} \cdot \dfrac{t^n}{n!}$

$$\left(\operatorname{vol}(X_t) = 2^{r_2} \cdot \underbrace{\operatorname{vol}_{Leb.}(\varphi(X_t))} \right.$$

$$\mathbb{R}^{r_1+2r_2} \ni (x_1, \dots, x_{r_1}, y_1, z_1, \dots, y_{r_2}, z_{r_2})$$

$$\left. |x_1| + \dots + |x_{r_1}| + 2\sqrt{y_1^2+z_1^2} + \dots + 2\sqrt{y_{r_2}^2+z_{r_2}^2} \leq t. \right)$$

Teorema Dado $I \subseteq \mathcal{O}_K$ no nulo, existe $\alpha \in I$, no nulo, t.q.

$$|N_{K/\mathbb{Q}}(\alpha)| \leq M_K \cdot N_{K/\mathbb{Q}}(I), \text{ donde}$$

$$M_K = \frac{n!}{n^n} \cdot \left(\frac{4}{\pi}\right)^{r_2} \cdot \sqrt{|\Delta_K|} \text{ — cota de Minkowski}$$

Demostración $\Lambda = \varphi(I) \subset K_{\mathbb{R}}$

$X_t \subset K_{\mathbb{R}}$ $\quad \boxed{\operatorname{vol}(X_t) = 2^n \cdot \operatorname{covol} \Lambda}$

$\exists\ \alpha \in I$, $\alpha \neq 0$ t.q. $\varphi(\alpha) \in X_t$.

$$\boxed{\sum_\tau |\tau(\alpha)| \leq t.}$$

$$2^{r_1} \cdot \pi^{r_2} \cdot \frac{t^n}{n!} = 2^{r_1+2r_2} \cdot \sqrt{|\Delta_K|} \cdot N_{K/\mathbb{Q}}(I)$$

$$t^n = n! \left(\frac{4}{\pi}\right)^{r_2} \cdot \sqrt{|\Delta_K|} \cdot N_{K/\mathbb{Q}}(I)$$

$$|N_{K/\mathbb{Q}}(\alpha)| = \prod_\tau |\tau(\alpha)| \leq \frac{1}{n^n} \cdot \sum_\tau |\tau(\alpha)|$$

$$\leq \frac{1}{n^n} \cdot t^n = \boxed{\frac{n!}{n^n} \cdot \left(\frac{4}{\pi}\right)^{r_2} \cdot \sqrt{|\Delta_K|}} \cdot N_{K/\mathbb{Q}}(I)$$

$\underbrace{}_{M_K}$ M_K.

$$\frac{1}{n}\sum_{\tau} |\tau(\alpha)| \geq \left(\prod_{\tau} |\tau(\alpha)|\right)^{1/n}$$

$\boxtimes$

$$I = \mathcal{O}_K \Rightarrow 1 \leq |N_{K/\mathbb{Q}}(\alpha)| \leq M_K$$

$\boxed{r_1 + 2r_2 = n}$

$$\Leftrightarrow 1 \leq \frac{n!}{n^n} \cdot \left(\frac{4}{\pi}\right)^{r_2} \cdot \sqrt{|\Delta_K|}.$$

$$|\Delta_K| \geq \left(\frac{n^n}{n!}\right)^2 \cdot \left(\frac{\pi}{4}\right)^{2r_2} \geq \left(\frac{n^n}{n!}\right)^2 \cdot \left(\frac{\pi}{4}\right)^{n}.$$

$n = [K:\mathbb{Q}]$. $\boxed{n=1}$ $K = \mathbb{Q}$. $1 \geq \frac{\pi}{4}$.

$\boxed{n=2}$ $|\Delta_K| \geq \frac{\pi^2}{4} > 1$

Teorema (Minkowski) Si $n > 1$ (es decir, $K \neq \mathbb{Q}$) entonces, $|\Delta_K| > 1$.

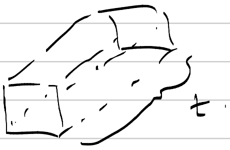
En particular, algún primo siempre se ramifica en K.

§ Teorema de Hermite

Lema Para todo $K/\mathbb{Q}$ existe $\alpha \in \mathcal{O}_K$ t.q. $K = \mathbb{Q}(\alpha)$ y $\forall\, \tau : K \hookrightarrow \mathbb{C}$ $|\tau(\alpha)| \leq C$, donde C depende de Δ_K.

Dem 1) Si K tiene encaje real, $\rho : K \hookrightarrow \mathbb{R}$.

$t > 0$ $\quad X_t = \{ (x_\tau)_\tau \in K_{\mathbb{R}} \mid |x_\rho| < t,\ |x_\tau| < 1\ \ \tau \neq \rho \}$



2) Si K no tiene encajes reales,

$\sigma, \bar\sigma : K \hookrightarrow \mathbb{C}$

X_t : $x_\sigma, x_{\bar\sigma} \in (-1,+1) + (-t,+t)i$,

$|x_\tau| < 1$ para $\tau \neq \sigma, \bar\sigma$.

Tomemos t t.q. $\boxed{\mathrm{vol}(X_t) > 2^n \cdot \sqrt{|\Delta_K|}}$

Minkowski: $\exists\, \alpha \in \mathcal{O}_K$, $\alpha \neq 0$ t.q. $\Phi(\alpha) \in X_t$

Nos gustaría ver que $K = \mathbb{Q}(\alpha)$.

$$\begin{tikzcd} K \arrow[r, hook] & \mathbb{C} \\ \mathbb{Q}(\alpha) \arrow[u, dash] \arrow[ur, hook] & \end{tikzcd}$$

todo encaje $\mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$ se levanta a $[K:\mathbb{Q}(\alpha)]$ encajes de $K \hookrightarrow \mathbb{C}$

$\forall\, \tau_1, \tau_2 : K \hookrightarrow \mathbb{C}$ $\quad \tau_1 \neq \tau_2 \Rightarrow \tau_1(\alpha) \neq \tau_2(\alpha)$.

$|N_{K/\mathbb{Q}}(\alpha)| = \prod_\tau |\tau(\alpha)| < 1$ contradicción.

$\alpha \in \mathcal{O}_K$, $\alpha \neq 0 \Rightarrow |N_{K/\mathbb{Q}}(\alpha)| \in \mathbb{Z}_{\geq 1}$.

$|\tau(\alpha)| \leq \text{algo}(\Delta_K)$ $\qquad \otimes$

Teorema (Hermite) $\forall\, C > 0$, salvo iso, existe un # finito de campos de números $K/\mathbb{Q}$ con $|\Delta_K| < C$.

Dem. Cota de Minkowski $\Rightarrow$ basta suponer que $|\Delta_K| = C$ es fijo y $n = [K:\mathbb{Q}]$ es fijo.

$K = \mathbb{Q}(\alpha)$, $\quad |\tau(\alpha)| \leq$ en términos de C

$$f_{\mathbb{Q}}^{\alpha} = \prod_{\tau} (x - \tau(\alpha)) \in \mathbb{Z}[x].$$

$\Rightarrow$ los coef. de $f_{\mathbb{Q}}^{\alpha}$ están acotados.

$\Rightarrow$ hay un # finito de posibilidades. $\boxtimes$

Ejemplo

Campos cúbicos con $|\Delta_K| \leq 100$.

$x^3 + x^2 - 2x - 1 \qquad \Delta_K = 49$

$x^3 - 3x - 1 \qquad \Delta = 81.$

$N(C)$ = el # de campos cúbicos con el $|\Delta_K| \leq C$.

$$\lim_{C \to \infty} \frac{N(C)}{C} = \begin{cases} \dfrac{1}{12 \cdot \zeta(3)}, & \text{para cúbicos reales} \\ \dfrac{1}{4\zeta(3)}, & \text{cúbicos complejos.} \end{cases}$$

($\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$ es la ζ de Riemann)