$K = \mathbb{Q}(\sqrt{d})$ $\qquad \mathbb{Z}[\sqrt{d}]^\times \subset \mathcal{O}_K^\times \cong \{\pm 1\} \times \langle u \rangle$

$$\begin{matrix} \mathbb{Z}[\sqrt{d}]^\times \\ \wr\wr \\ \{\pm 1\} \times \langle v \rangle \end{matrix}$$

$x^2 + dy^2 = \pm 1.$

## § Fracciones continuas.

$$[a_0] = a_0$$
$$[a_0, a_1] = a_0 + \frac{1}{a_1}$$
$$[a_0, a_1, a_2] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}}$$

$$[a_0, \dots, a_n] = a_0 + \frac{1}{[a_1, \dots, a_n]}$$

**Def** Sea $a_0, a_1, a_2, \dots$ sucesión de $a_n \in \mathbb{Z}$, $a_n \geq 1$ para $n \geq 1$.

$$[a_0, a_1, a_2, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

El valor correspondiente es

$$\lim_{n \to \infty} x_n, \qquad x_n = [a_0, \dots, a_n].$$

**Lema** Definamos

- $p_{-2} = 0$, $p_{-1} = 1$, $\quad p_n = a_n p_{n-1} + p_{n-2}$. $\qquad p_0 = a_0$
- $q_{-2} = 1$, $q_{-1} = 0$, $\quad q_n = a_n q_{n-1} + q_{n-2}$. $\qquad q_0 = 1$

1) $\forall\, \alpha > 0\ \forall\, n \geq 1$

$$[a_0, \dots, a_{n-1}, \alpha] = \frac{\alpha p_{n-1} + p_{n-2}}{\alpha q_{n-1} + q_{n-2}}.$$

$$x_n = [a_0, \dots, a_n] = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} = \frac{p_n}{q_n}$$

2) $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}$ (1) $\qquad x_n - x_{n-1} = \dfrac{(-1)^{n+1}}{q_n q_{n-1}}$ (2)

$p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n$ $\qquad x_n - x_{n-2} = \dfrac{(-1)^n a_n}{q_n q_{n-2}}$ (3)

(1) $\Rightarrow$ $\mathrm{mcd}(p_n, q_n) = 1$.

$$1 = q_0 \leq q_1 < q_2 < q_3 < \cdots$$

(2) $\Rightarrow$

$$\lim_{n \to \infty} (x_n - x_{n-1}) = 0.$$

$$x_0 < x_2 < x_4 < \cdots < x_1 \qquad (x_{2n})$$

$$x_1 > x_3 > x_5 > \cdots > x_0 \qquad (x_{2n+1})$$

Conclusión: existe $\lim_{n \to \infty} x_n$

Ejemplo $\alpha = [1, 1, 1, \dots] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}$

$$x_n = \frac{p_n}{q_n} = \frac{F_{n+2}}{F_{n+1}}. \qquad p_{-2} = 0, \; p_{-1} = 1, \; p_n = p_{n-1} + p_{n-2}$$

$$q_{-2} = 1, \; q_{-1} = 0, \; q_n = q_{n-1} + q_{n-2}$$

$$x_0 = 1, \quad x_1 = 2, \quad x_2 = \frac{3}{2}, \quad x_3 = \frac{5}{3}, \quad \dots$$

$$\alpha = [1, \alpha] = 1 + \frac{1}{\alpha} \iff \alpha^2 - \alpha - 1 = 0.$$

$$\alpha = \frac{1+\sqrt{5}}{2} = \lim_{n \to \infty} \frac{F_{n+2}}{F_{n+1}}.$$

Proposición $\alpha = [a_0, a_1, \dots]$ es irracional y está definido de modo único por los $a_n$.

Dem

$$x_0 < \alpha < x_1 \qquad a_0 < \alpha < a_0 + \frac{1}{a_1}$$

$$\underline{a_1 \geq 1} \implies \lfloor \alpha \rfloor = a_0$$

$$\alpha = a_0 + \frac{1}{[a_1, a_2, \dots]}$$

$$[a_0, a_1, \dots] = [b_0, b_1, \dots] \iff a_n = b_n \quad \forall n.$$

$x_n = \frac{p_n}{q_n}$ $\qquad x_n < \alpha < x_{n+1}$ para $n$ par.

$$0 < |\alpha - x_n| < |x_{n+1} - x_n| = \frac{1}{q_{n+1} q_n}$$

$$0 < |\alpha q_n - p_n| < \frac{1}{q_{n+1}} \qquad \text{si } \alpha = \frac{a}{b}$$

$$0 < |\underbrace{a q_n - b p_n}| < \underbrace{\frac{b}{q_{n+1}}} < 1 \quad \text{para } n \gg 0.$$

Contradicción $\square$

§ Fracción continua asociada a un $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

$$\alpha_0 = \alpha \qquad a_0 = \lfloor \alpha \rfloor$$

$$\underbrace{\alpha_n} = \frac{1}{\alpha_{n-1} - a_{n-1}}, \quad \text{donde } \underbrace{a_{n-1}} = \lfloor \alpha_{n-1} \rfloor$$

$$0 < \alpha_{n-1} - a_{n-1} < 1 \Rightarrow a_n \geq 1 \quad \text{para } n \geq 1.$$

$\Longrightarrow$ $[a_0, a_1, a_2, \ldots]$ fracción continua

$$\alpha = [a_0, \alpha_1] = [a_0, a_1, \alpha_2] = \cdots = [a_0, a_1, \ldots, a_{n-1}, \alpha_n].$$

$$\alpha = \frac{\alpha_n p_{n-1} + p_{n-2}}{\alpha_n q_{n-1} + q_{n-2}}.$$

$$\lim_{n \to \infty} \left( \frac{\alpha_n p_{n-1} + p_{n-2}}{\alpha_n q_{n-1} + q_{n-2}} - \frac{p_n}{q_n} \right) = 0.$$

fracciones continuas infinitas $[a_0, a_1, a_2, \ldots]$ $\longrightarrow$ $\mathbb{R} \setminus \mathbb{Q}$ $\longleftarrow$

**Ejemplo** $\alpha = \pi$

$\alpha_0 = \alpha = \pi$ $\qquad a_0 = \lfloor \alpha_0 \rfloor = 3$

$\alpha_1 = \dfrac{1}{\alpha_0 - a_0} = 7{,}06\ldots$ $\qquad a_1 = 7.$

$\alpha_2 = \dfrac{1}{\alpha_1 - a_1} = 15{,}55$ $\qquad a_2 = 15.$

$\alpha_3 = \dfrac{1}{\alpha_2 - a_2} = 1{,}003$ $\qquad a_3 = 1.$

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}$$

$3 + \frac{1}{7} = \boxed{\frac{22}{7}}$

$$3 + \cfrac{1}{7 + \cfrac{1}{15 + 1}} = \frac{355}{113} = \underline{3{,}141552}9$$

**Ejemplo** (Euler) $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, \ldots]$

$\Longrightarrow e \notin \mathbb{Q}.$

§ **Aproximaciones de $\alpha$ por $\frac{p_n}{q_n}$**

$$\left|\alpha - \frac{a}{b}\right| < \left|\alpha - \frac{p_n}{q_n}\right| \Longrightarrow b > q_n.$$

$(q_{n+1} > q_n).$

**Proposición** Si $\underline{|\alpha b - a| < |\alpha q_n - p_n|} \Rightarrow b \geq q_{n+1}$

**Dem** Supongamos que
$|\alpha b - a| < |\alpha q_n - p_n|$ y $b < q_{n+1}.$

$$\begin{cases} \underline{x q_n + y q_{n+1}} = b \\ \underline{x p_n + y p_{n+1}} = a \end{cases} \qquad \det \begin{pmatrix} q_n & q_{n+1} \\ p_n & p_{n+1} \end{pmatrix} = \pm 1$$

$\exists$ solución $x, y \in \mathbb{Z}$, $(x,y) \neq (0,0)$

De hecho, $x \neq 0$, $y \neq 0$, y además, $xy < 0$.

$$\alpha b - a = \underline{x} (\underline{\alpha q_n - p_n}) + \underline{y} (\underline{\alpha q_{n+1} - p_{n+1}}).$$

$$|\alpha b - a| = |x| \cdot |\alpha q_n - p_n| + |y| \cdot |\alpha q_{n+1} - p_{n+1}|$$

$$> |\alpha q_n - p_n|$$

$\boxtimes$

**Corolario** $\left|\alpha - \frac{a}{b}\right| < \frac{1}{2b^2}$, entonces $\underbrace{\frac{a}{b} = \frac{p_n}{q_n}}$

para algún $n$.

**Dem.** $q_n \leq b < q_{n+1}$ para algún $n$

$$|\alpha q_n - p_n| \leq |\alpha b - a| < \frac{1}{2b}$$

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{2bq_n}$$

$$\left|\frac{a}{b} - \frac{p_n}{q_n}\right| \leq \left|\alpha - \frac{a}{b}\right| + \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{2b^2} + \frac{1}{2bq_n}$$

Si $\frac{a}{b} \neq \frac{p_n}{q_n}$, entonces

$$\left|\frac{a}{b} - \frac{p_n}{q_n}\right| = \frac{|aq_n - bp_n|}{bq_n} \geq \frac{1}{bq_n}.$$

$$\left.\frac{1}{bq_n} < \frac{1}{2b^2} + \frac{1}{2bq_n}\right] \Rightarrow \underline{b < q_n}$$

no es el caso

$\boxtimes$

## § Fracciones continuas periódicas

**def** ·) $\alpha = [a_0, a_1, a_2, \dots]$ es **periódica**

si $\exists$ $n_0$, $k \geq 1$ t.q. $a_n = a_{n+k}$ $\forall n \geq n_0$.

**Notación**: $[a_0, a_1, \dots, a_{n_0-1}, \overline{a_{n_0}, \dots, a_{n_0+k-1}}]$.

·) si $n_0 = 0 \Rightarrow \alpha = [\overline{a_0, \dots, a_{k-1}}]$

$\Rightarrow$ **puramente periódica**.

**Proposición** La fracción continua para $\alpha$ es periódica $\Rightarrow [\mathbb{Q}(\alpha) : \mathbb{Q}] = 2$.

**Dem** ·) $\alpha = [\overline{a_0, \dots, a_{k-1}}]$

$$\alpha = [a_0, \dots, a_{k-1}, \alpha] = \frac{\alpha p_{k-1} + p_{k-2}}{\alpha q_{k-1} + q_{k-2}}.$$

$$\Rightarrow q_{k-1}\alpha^2 + (q_{k-2} - p_{k-1})\alpha - p_{k-2} = 0.$$

$[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2$.

parte periódica

·) si $\alpha = [a_0, \dots, a_{k-1}, \beta]$

puramente periódico.

$$= \frac{\beta \cdot p_{k-1} + p_{k-2}}{\beta \cdot q_{k-1} + q_{k-2}} \in \mathbb{Q}(\beta).$$

$\square$

**Ejemplo** $\alpha = [\overline{1, 2, 3}] = [1, 2, 3, \alpha]$

$$\alpha = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{\alpha}}} \Leftrightarrow 7\alpha^2 - 8\alpha - 3 = 0$$

$$\alpha = \frac{4 + \sqrt{37}}{7}.$$

**Teorema** (Lagrange) Si $[\mathbb{Q}(\alpha):\mathbb{Q}]=2$ entonces la fracción continua para $\alpha$ es periódica.

**Ejemplo** $\sqrt{11} = [3, \overline{3, 6}]$ ← ejercicio.

$\alpha_3 = \alpha_1$

$a_3 = a_1$.