11/11 # FUNCIÓN ZETA DE DEDEKIND

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

Lema ◦) $\zeta(s)$ converge absolutamente si $\operatorname{Re} s > 1$

◦) $\displaystyle\lim_{s \to 1^+} (s-1)\zeta(s) = 1.$

Dem $\left|\frac{1}{n^s}\right| = \frac{1}{n^{\operatorname{Re} s}}$ $\qquad$ $s > 1$

$$\int_n^{n+1} \frac{dx}{x^s} \leq \frac{1}{n^s} \leq \int_{n-1}^{n} \frac{dx}{x^s}$$

$$\int_1^{\infty} \frac{dx}{x^s} \leq \zeta(s) \leq 1 + \int_1^{\infty} \frac{dx}{x^s}$$

$$\frac{1}{s-1} \leq \zeta(s) \leq \frac{s}{s-1}$$

$$1 \leq (s-1)\zeta(s) \leq s$$

$$\Rightarrow \lim_{s \to 1^+} (s-1)\zeta(s) = 1. \qquad \boxtimes$$

def $K/\mathbb{Q} \rightsquigarrow$ función zeta de Dedekind

$$\zeta_K(s) = \sum_{0 \neq I \in \mathcal{O}_K}' \frac{1}{N_{K/\mathbb{Q}}(I)^s}$$

$K = \mathbb{Q}$: $\mathcal{O}_K = \mathbb{Z}$, $\qquad n\mathbb{Z}$, $\quad n = 1, 2, 3, \ldots$

$$\zeta_{\mathbb{Q}}(s) = \zeta(s).$$

Proposición ◦) $\zeta_K(s)$ converge absolutamente para $\operatorname{Re} s > 1$.

◦) $$\zeta_K(s) = \prod_{\mathfrak{p} \subset \mathcal{O}_K} \frac{1}{1 - N_{K/\mathbb{Q}}(\mathfrak{p})^{-s}} \qquad (*)$$

Dem Será suficiente probar que el producto $(*)$ converge absolutamente para $s > 1$.

$$\prod_{\mathfrak{p}} \frac{1}{1-N_{K/\mathbb{Q}}(\mathfrak{p})^{-s}} = \prod_{\mathfrak{p}} \sum_{e\geq 1}' \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{p}^e)^s} = \sum_{I\neq 0}' \frac{1}{N_{K/\mathbb{Q}}(I)^s} = \zeta_K(s)$$

$I = \mathfrak{p}_1^{e_1}\cdots\mathfrak{p}_s^{e_s}$

$$\prod_{n\geq 1}(1+|x_n|) \text{ converge} \iff \sum_{n\geq 1}|x_n| \text{ converge.}$$

$$\sum_{\mathfrak{p}} \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{p})^s} = \sum_p' \sum_{\mathfrak{p}\mid p} \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{p})^s} \leq \sum_p' \frac{[K:\mathbb{Q}]}{p^s} < [K:\mathbb{Q}]\cdot\zeta(s)$$

Dado $p\in\mathbb{Z}$, hay a lo sumo $[K:\mathbb{Q}]$ ideales primos $\mathfrak{p}$ t.q. $\mathfrak{p}\mid p$

$$p\mathcal{O}_K = \mathfrak{p}_1^{e_1}\cdots\mathfrak{p}_r^{e_r} \qquad \underbrace{\sum_i e_i f_i = [K:\mathbb{Q}]}$$

$$N(\mathfrak{p}) = p^f$$

$\boxtimes$

Ejemplo: $$\zeta_{\mathbb{Q}(i)}(s) = \prod_{\mathfrak{p}\subset\mathbb{Z}[i]} \frac{1}{1-N_{K/\mathbb{Q}}(\mathfrak{p})^{-s}}$$

- $p=2$ se ramifica $\mathfrak{p}_2$ $\quad N(\mathfrak{p}_2)=2$.
- $p\equiv 1 \pmod 4$ $\quad p\mathbb{Z}[i] = \mathfrak{p}\cdot\bar{\mathfrak{p}}$
  $N(\mathfrak{p}) = N(\bar{\mathfrak{p}}) = p$.
- $p\equiv 3 \pmod 4$ $\quad p\mathbb{Z}[i] = \mathfrak{p}$ ideal primo.
  $N(\mathfrak{p}) = p^2$

$$\zeta_{\mathbb{Q}(i)}(s) = \frac{1}{1-2^{-s}} \cdot \prod_{p\equiv 1\,(4)} \frac{1}{(1-p^{-s})^2} \prod_{p\equiv 3\,(4)} \frac{1}{1-p^{-2s}}$$

$$\frac{1}{1-p^{-2s}} = \underbrace{\frac{1}{1-p^{-s}}}\cdot\frac{1}{1+p^{-s}}$$

$$\zeta_{\mathbb{Q}(i)}(s) = \boxed{\zeta(s)} \prod_p \frac{1}{1-\chi(p)\,p^{-s}}.$$

$$\chi(p) = \begin{cases} +1, & p\equiv 1 \quad (4) \\ -1, & p\equiv 3 \quad (4). \end{cases}$$ $\rightsquigarrow$ carácter de Dirichlet mód 4.

$$L(s,\chi) = \prod_p \frac{1}{1-\chi(p)\,p^{-s}} = \sum_{n\geq 1}{}' \frac{\chi(n)}{n^s}$$

$$\boxed{\zeta_{\mathbb{Q}(i)}(s) = \zeta(s)\cdot L(s,\chi)}$$

$\mathbb{Z}[i]$ es un DIP. $\quad \mathbb{Z}[i]^\times = \{\pm 1, \pm i\}$.

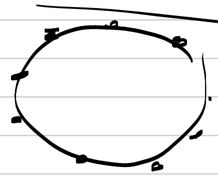
$$\zeta_{\mathbb{Q}(i)}(s) = \sum_{\emptyset \neq I \subset \mathbb{Z}[i]} \frac{1}{N(I)^s}$$

$$I = (\alpha) = (x+yi) \qquad N(I) = N(\alpha) = x^2+y^2$$

$$C(n) = \#\{(x,y)\in\mathbb{Z}^2 \mid x,y>0,\ x^2+y^2=n\}$$

$$\zeta_{\mathbb{Q}(i)}(s) = \sum_{n\geq 1}{}' \frac{C(n)}{n^s} = \left(\sum_{n\geq 1} \frac{1}{n^s}\right)\cdot\left(\sum_{n\geq 1}{}' \frac{\chi(n)}{n^s}\right)$$

$$C(n) = \sum_{d\mid n} \chi(d)$$



$4\,C(n)$

mcd $(m,n) \Rightarrow \chi(mn) = \chi(m)\cdot\chi(n)$.

$$C(mn) = C(m)\cdot C(n)$$

$$C(p^e) = \begin{cases} 0, & \text{si } p\equiv 3\ (4),\ e \text{ impar.} \\ 1, & \text{si } p\equiv 3\ (4),\ e \text{ par.} \\ e+1, & \text{si } p\equiv 1\ (4). \end{cases}$$

Teorema $\quad n = p_1^{e_1}\cdots p_s^{e_s}$ (generalización del t. de Fermat sobre $p = x^2+y^2$)

- •) $p_i \equiv 3\ (4)$ y e_i es impar para algún i

  $\Rightarrow$ n no tiene forma x^2+y^2

- •) En el caso contrario,

$$C(n) = \prod_{p_i\equiv 1\ (4)} (e_i+1)$$

el número de repr. de n como x^2+y^2, con $x,y>0$ es $C(n)$.

Ejemplo $n = 25 = 5^2 = 4^2 + 3^2$

Comentario Si $K/\mathbb{Q}$ abeliana | $K \hookrightarrow \mathbb{Q}(\zeta_n)$

$$\zeta_K(s) = \prod_{\chi} L(s,\chi)$$

[Washington, cyclotomic fields]

$\chi = 1 \Rightarrow L(s,\chi) = \zeta(s)$.

§ Fórmula analítica del número de clases.

encajes $K \hookrightarrow \mathbb{R}$.

pares de encajes complejos $K \hookrightarrow \mathbb{C}$.

$$\lim_{s\to 1^+} (s-1)\cdot \zeta_K(s) = \frac{2^{r_1}\cdot (2\pi)^{r_2}\cdot \mathrm{Reg}_K \cdot h_K}{\# \mu_K \cdot \sqrt{|\Delta_K|}}$$

regulador.

$\# Cl(K)$

$(\mathcal{O}_K^\times)_{\text{tors}}$

discriminante

$\sqrt{|\Delta_K|}$ = covol de $\mathcal{O}_K$ como retículo.

$$\Phi : K \hookrightarrow K_{\mathbb{R}} \qquad \Phi(\mathcal{O}_K) = \Lambda$$

Reg_K = $\left(\begin{array}{c}\text{salvo una}\\ \text{normalización}\end{array}\right)$ covol. de $\mathcal{O}_K^\times$ como retículo.

§ Regulador. Recordatorio.

Consideremos diferentes encajes $K \hookrightarrow \mathbb{C}$.

$$\underbrace{\sigma_1, \dots, \sigma_{r_1}}_{\text{reales}},\ \underbrace{\sigma_{r_1+1}, \overline{\sigma_{r_1+1}}, \dots, \sigma_{r_1+r_2}, \overline{\sigma_{r_1+r_2}}}_{\text{complejos}}.$$

$$\Phi : K^\times \longrightarrow K_{\mathbb{R}}^\times$$
$$\alpha \longmapsto (\sigma_i(\alpha))_i.$$

$$\boxed{K_{\mathbb{C}} \circlearrowleft G_{\mathbb{R}} \qquad K_{\mathbb{C}}^{G_{\mathbb{R}}} = K_{\mathbb{R}}}$$

$$\ell : K_{\mathbb{R}}^{\times} \longrightarrow \mathbb{R}^{r_1+r_2}$$

$$(z_{\sigma_i})_i \longmapsto \left(n_i \cdot \log |z_{\sigma_i}|\right)_i \qquad n_i = \begin{cases} 1, & \sigma_i \text{ real} \\ 2, & \sigma_i \text{ complejo} \end{cases}$$

$$\begin{tikzcd}
\mathcal{O}_K^{\times} \arrow[r, "L"] \arrow[d, hook] & & H = \{(x_i) \mid \sum_i' x_i = 0\} \arrow[d, hook] \\
K^{\times} \arrow[r, hook, "\Phi"] \arrow[d, "N"] & K_{\mathbb{R}}^{\times} \arrow[r, "\ell"] & \mathbb{R}^{r_1+r_2} \arrow[d, "\sum'"] \\
\mathbb{Q} \arrow[rr, "\log|\cdot|"] & & \mathbb{R}.
\end{tikzcd}$$

$L(\mathcal{O}_K^{\times})$ es un retículo de rango completo en H

$\dim_{\mathbb{R}} H = r_1 + r_2 - 1.$

$u_1, \dots, u_{r_1+r_2-1} \in \mathcal{O}_K^{\times}$ — unidades fundamentales

$L(u_1), \dots, L(u_{r_1+r_2-1})$ — una base de H.

$$L = \frac{1}{\sqrt{r_1+r_2}} \underbrace{(1, \dots, 1)}_{r_1+r_2} \in \mathbb{R}^{r_1+r_2} \quad \text{ortogonal a } H.$$

$L, \boxed{L(u_1), \dots, L(u_{r_1+r_2-1})}$ — una base de $\mathbb{R}^{r_1+r_2}$

$$\operatorname{covol}\left(L(\mathcal{O}_K^{\times})\right) = \pm \det \begin{pmatrix} L_1 & L_1(u_1) & \cdots & L_1(u_{r_1+r_2-1}) \\ L_2 & L_2(u_1) & \cdots & L_2(u_{r_1+r_2-1}) \\ \vdots & \vdots & & \vdots \\ L_{r_1+r_2} & L_{r_1+r_2}(u_1) & \cdots & L_{r_1+r_2}(u_{r_1+r_2-1}) \end{pmatrix}$$

Sumamos todas las filas a la i-ésima fila,

i: $(\sqrt{r_1+r_2}, 0, 0, \dots, 0)$

Covol $L(\mathcal{O}_K^\times) = \sqrt{r_1+r_2} \cdot \underline{Reg_K}$

Reg_K = el valor absoluto del det. de culquier menor de rango r_1+r_2-1 de la matriz

$$\Big(L_i(u_j) \Big)_{\substack{1\le i\le r_1+r_2 \\ 1\le j\le r_1+r_2-1}} = \big(n_i \log |\sigma_i(u_j)| \big).$$

Ejemplo $K = \mathbb{Q}(\sqrt{d})$.

$u > 1$.

$\mathcal{O}_K^\times = \{\pm 1\} \times \langle u \rangle$.

$\sigma_1 : \sqrt{d} \longmapsto \sqrt{d}$

$\sigma_2 : \sqrt{d} \longmapsto -\sqrt{d}$.

$$\begin{pmatrix} \log|\sigma_1(u)| \\ \log|\sigma_2(u)| \end{pmatrix}$$

$Reg_K = \log u$

Ejemplo $K = \mathbb{Q}(\zeta_7)$

$$\begin{array}{l} u_1 = 1+\zeta_7 \\ u_2 = \zeta_7 + \zeta_7^4 \end{array}$$

$r_1 = 0$, $\underline{r_2 = 3}$.

$\sigma_1 = \zeta_7 \mapsto \zeta_7$

$\sigma_2 = \zeta_7 \longmapsto \zeta_7^2$

$\sigma_3 : \zeta_7 \longmapsto \zeta_7^3$

$$Reg_K = \pm \det \begin{pmatrix} 2\cdot\log|1+\zeta_7| & 2\cdot\log|\zeta_7+\zeta_7^4| \\ 2\cdot\log|1+\zeta_7^2| & 2\cdot\log|\zeta_7^2+\zeta_7| \end{pmatrix}$$

$= 2{,}101818\ldots$

La próxima clase:
(18 de noviembre)

$K = \mathbb{Q}(\sqrt{-p})$

$p \equiv 3 \ (8)$

$$h_K = \frac{1}{3} \sum_{1 \le a \le \frac{p-1}{2}} \left(\frac{a}{p}\right)$$