# Planes para el resto del curso.

**hoy**: aplicaciones de la f. del # de clases

$$\lim_{s\to 1^+} (s-1)\,\zeta_K(s) \rightsquigarrow h_K$$

**clase 27**: Prueba de la f. del # de clases.

**clase 28**: $\zeta_K(s) = \prod_{\chi} L(s,\chi)$

$K/\mathbb{Q}$ abeliana

"valores especiales"

**clase 29**: $n \in \mathbb{Z}$, $\zeta_K(n) = \prod_{\chi} L(n,\chi)$.

$$\zeta(2) = \frac{\pi^2}{6}$$

**clase 30** $\zeta_K(s) = \zeta_{K'}(s)$ "equivalencia aritmética"

$$\lim_{s\to 1^+} (s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2}\, Reg_K}{\# \mu_K \cdot \sqrt{|\Delta_K|}} h_K.$$

§ El uso de la fórmula del # de clases

Ejemplo $K = \mathbb{Q}(\sqrt{5})$

$$\zeta_K(s) = \zeta(s)\cdot L_1(s,\chi) \qquad \chi(n) = \left(\frac{n}{5}\right)$$

$$\lim_{s\to 1^+} (s-1)\zeta_K(s) = L_1(1,\chi)$$

Ejercicio: $\chi(n) = \left(\frac{n}{p}\right)$ $\qquad g(\chi) = \sum_{1\leq a\leq p-1} \chi(a)\cdot\zeta_p^a$

$$\exp\left(g(\chi)\cdot L_1(1,\chi)\right) = \prod_{\substack{n\\ n\not\equiv \square (p)}} (1-\zeta_p^n)\cdot \prod_{\substack{r\\ r\equiv\square (p)}} (1-\zeta_p^r)^{-1}$$

$$g(\chi) = \left(\frac{1}{5}\right)\zeta_5 + \left(\frac{2}{5}\right)\zeta_5^2 + \left(\frac{3}{5}\right)\zeta_5^3 + \left(\frac{4}{5}\right)\zeta_5^4$$

$$= \zeta_5 - \zeta_5^2 - \zeta_5^2 + \zeta_5^4 = \sqrt{5}.$$

$$\boxed{\mathbb{Q}(\zeta_5)} \quad (1-\zeta_5^2)(1-\zeta_5^3)\cdot(1-\zeta_5)^{-1}(1-\zeta_5^4)^{-1} = \frac{3+\sqrt{5}}{2}.$$

$$L_1(1,\chi) = \frac{1}{\sqrt{5}}\cdot\log\frac{3+\sqrt{5}}{2}.$$

$$\|$$

$$\frac{2^{r_1}(2\pi)^{r_2}\cdot Reg_K}{\#\mu_K\cdot\sqrt{|\Delta_K|}}\cdot h_K = \frac{2}{\sqrt{5}}\cdot\log\left(\frac{1+\sqrt{5}}{2}\right)\cdot h_K.$$

$r_1 = 2$
$r_2 = 0.$
$\mu_K = \{\pm 1\}$
$\Delta_K = 5.$
$u = \frac{1+\sqrt{5}}{2}.$

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2}$$

$h_K = 1.$ $\qquad Cl(\mathbb{Q}(\sqrt{5})) = 1.$

## Ejemplo

$K = \mathbb{Q}(\sqrt{10})$ $\qquad u = 3+\sqrt{10}$

$$\lim_{s\to 1^+} (s-1)\cdot\zeta_K(s) = \frac{2}{\sqrt{40}}\cdot \log(3+\sqrt{10})\cdot h_K.$$

$\| \quad 1,15\ldots \quad \Longrightarrow h_K = 2$

$\zeta_K(s) = \zeta(s)\cdot L(s,\chi)$. $\qquad L(1,\chi)$

[Borevich, Shafarevich. "Number Theory"]

{ **Número de clases de $\mathbb{Q}(\sqrt{-p})$**, $p\equiv 3\ (4)$.

$$\lim_{s\to 1^+}(s-1)\zeta_K(s) = \frac{2^{r_1}\cdot(2\pi)^{r_2}\cdot \mathrm{Reg}_K\cdot h_K}{\#\mu_K\cdot\sqrt{|\Delta_K|}} \qquad \Big|\ p>3.$$

$\mathcal{O}_K^\times = \mu_K = \{\pm 1\}$ $\qquad \Delta_K = -p.$ $\qquad \mathrm{Reg}_K = 1.$

$r_1 = 0 \qquad r_2 = 1$

$$\lim_{s\to 1^+}(s-1)\zeta_K(s) = \frac{\pi}{\sqrt{p}}\cdot h_K$$

$\zeta_K(s) = \zeta(s)\cdot L(s,\chi).$

$$L(1,\chi) = \frac{\pi}{\sqrt{p}}\cdot h_K.$$

$p\equiv 3\ (4)$. $\left(\frac{-p}{q}\right) = \left(\frac{q}{p}\right)$

$\chi(n) = \left(\frac{n}{p}\right)$

$$h_K = \frac{\sqrt{p}}{\pi} L(1,\chi)$$

$$\exp(g(\chi)\cdot L(1,\chi)) = \prod_n (1-\zeta_p^n)\cdot \prod_r (1-\zeta_p^r)^{-1}$$

$$g(\chi) = \begin{cases} \sqrt{p}, & p\equiv 1\ (4) \\ i\sqrt{p}, & p\equiv 3\ (4). \end{cases} \qquad g(\chi)^2 = (-1)^{\frac{p-1}{2}}\cdot p$$

$$L(1,\chi) = -\frac{1}{i\sqrt{p}} \sum_{1 \le a \le p-1} \chi(a) \cdot \log(1-\zeta_p^a)$$

$$S_\chi = -\sum_a \chi(a) \cdot \log(1-\zeta_p^a) \qquad p \equiv 3 \ (4).$$

$$\chi(-1) = \left(\frac{-1}{p}\right) = -1.$$

$$\underline{2 S_\chi} = \sum_a \chi(a)\left(\log(1-\zeta_p^{-a}) - \log(1-\zeta_p^a)\right).$$

$$= \sum_a \chi(a) \cdot \log \frac{1-\zeta_p^{-a}}{1-\zeta_p^a} \qquad \chi(-a) = \underbrace{\chi(-1)}_{=-1} \cdot \chi(a)$$

$$= \sum_a \chi(a) \cdot \log\left(-\zeta_p^{-a}\right)$$

$$= \sum_a \chi(a) \cdot \log\left(\exp\left(\pi i - \frac{2\pi i a}{p}\right)\right)$$

$$= \sum_a \chi(a) \cdot \left(\pi i - \frac{2\pi i a}{p}\right)$$

$$= \sum_a \chi(a) \cdot 2\pi i\left(\frac{1}{2} - \frac{a}{p}\right)$$

$$S_\chi = \sum_a \chi(a) \cdot \pi i\left(\frac{1}{2} - \frac{a}{p}\right)$$

$$= \frac{\pi i}{p} \cdot \left( \frac{p}{2} \underbrace{\sum_a \chi(a)}_{=0} - \sum_a \chi(a) \cdot a \right)$$

$$S_\chi = -\frac{\pi i}{p} \cdot \sum_a \chi(a) \cdot a.$$

$$L(1,\delta) = \frac{1}{i\sqrt{p}} \cdot S_\chi = -\frac{\pi i}{i\sqrt{p}} \cdot \frac{1}{p} \cdot \sum_a \chi(a)\, a$$

$$h_K = \frac{\sqrt{p}}{\pi} \cdot L(1,\delta) = -\frac{1}{p} \cdot \sum_a \chi(a) \cdot a$$

**Lema** $p > 3$, $p \equiv 3\ (4)$. $\chi = \left(\frac{\cdot}{p}\right)$

$$\frac{1}{p} \cdot \sum_{1 \le a \le p-1} \chi(a) \cdot a = \begin{cases} -\frac{1}{3} \sum\limits_{1 \le a \le \frac{p-1}{2}} \chi(a), & p \equiv 3\ (8). \\ -\sum\limits_{1 \le a \le \frac{p-1}{2}} \chi(a) & p \equiv 7\ (8) \end{cases}$$

**Dem** $C = \frac{1}{p} \cdot \sum\limits_{1 \le a \le p-1} \left(\frac{a}{p}\right) \cdot a = ?$

$$\bullet)\ pC = \sum_{1 \le a \le p-1} \left(\frac{a}{p}\right) a = \sum_{1 \le a \le \frac{p-1}{2}} \chi(a) \cdot a + \sum_{1 \le a \le \frac{p-1}{2}} \chi(p-a) \cdot (p-a)$$

$p \equiv 3\ (4)$
$\chi(-1) = -1.$

$$= \sum_{1 \le a \le \frac{p-1}{2}} \chi(a) \cdot a - \sum_{1 \le a \le \frac{p-1}{2}} \chi(a) \cdot (p-a)$$

$$= 2 \cdot \sum_{1 \le a \le \frac{p-1}{2}} \chi(a) \cdot a - \boxed{p \cdot \sum_{1 \le a \le \frac{p-1}{2}} \chi(a)}.$$

$$\bullet)\ pC = \sum_{1 \le a \le p-1} \chi(a) \cdot a = \sum_{a \text{ par}}' \chi(a) \cdot a + \sum_{a \text{ impar}}' \chi(a) \cdot a.$$

$$= \sum_{1 \le a \le \frac{p-1}{2}}' \chi(2a) \cdot 2a + \sum_{a \text{ par}.} \chi(p-a) \cdot (p-a)$$

$$= \sum_{1 \le a \le \frac{p-1}{2}}' \chi(2a) \cdot 2a + \sum_{1 \le a \le \frac{p-1}{2}} \chi(-2a) \cdot (p-2a).$$

$$= 4\,\chi(2)\cdot \sum_{1\le a\le \frac{p-1}{2}} \chi(a)\cdot a \;-\; p\cdot\chi(2)\cdot \sum_{1\le a\le \frac{p-2}{2}} \chi(a)$$

$$p\cdot C = 2\cdot \sum_{1\le a\le \frac{p-1}{2}} \chi(a)\cdot a \;-\; p\cdot \sum_{1\le a\le \frac{p-1}{2}} \chi(a) \qquad (1)$$

$$p\cdot C = 4\,\chi(2)\cdot \sum_{1\le a\le \frac{p-1}{2}} \chi(a)\cdot a \;-\; p\cdot\chi(2)\cdot \sum_{1\le a\le \frac{p-2}{2}} \chi(a) \qquad (2).$$

$2\cdot\chi(2)\cdot(1) - (2)$

$$C\,(2\chi(2)-1) = -\chi(2)\cdot \sum_{1\le a\le \frac{p-1}{2}} \chi(a)$$

$$C = \frac{1}{p}\cdot \sum_{1\le a\le p-1} \left(\frac{a}{p}\right)\cdot a = -\frac{\chi(2)}{2\chi(2)-1} \sum_{1\le a\le \frac{p-1}{2}} \left(\frac{a}{p}\right)$$

$$\chi(2) = \left(\frac{2}{p}\right) = \begin{cases} -1, & p\equiv 3 \ (8) \\ +1, & p\equiv 7 \ (8) \end{cases} \qquad \square$$

**Teorema (Dirichlet)** $p>3$, $p\equiv 3\ (8)$.

$$h_{\mathbb{Q}(\sqrt{-p})} = \begin{cases} \frac{1}{3}\sum_{1\le a\le \frac{p-1}{2}} \left(\frac{a}{p}\right) & p\equiv 3\ (8) \\ \sum_{1\le a\le \frac{p-1}{2}} \left(\frac{a}{p}\right) & p\equiv 7\ (8) \end{cases}$$

**Corolario)** $p\equiv 3\ (4) \Rightarrow$ en $[1, \frac{p-1}{2}]$.

hay más cuadrados mód $p$ que no-cuadrados.

**Proposición** $\chi = \left(\frac{\cdot}{p}\right)$

$$\left| \sum_{m \leq a \leq n} \chi(a) \right| < \sqrt{p} \cdot \log p.$$

(Prueba en mis apuntes)

**Pólya-Vinogradov**: $\chi$ carácter mód $N$.

$$\left| \sum \chi(a) \right| = O(\sqrt{N} \cdot \log N).$$

**En gral,** $K = \mathbb{Q}(\sqrt{d})$

$$\zeta_K(s) = \zeta(s) \cdot L(s, \chi).$$

$h_K \longleftrightarrow L(1, \chi).$ (Borevich, Shafarevich, Capítulo 5)