23/11. (Bosquejo de) la prueba de la fórmula analítica del número de clases.

$$\lim_{s\to 1^+} (s-1)\,\zeta_K(s) = \boxed{\frac{2^{r_1}(2\pi)^{r_2}\,\mathrm{Reg}_K}{\#\mu_K\cdot\sqrt{|\Delta_K|}}}\; h_K$$

$$\zeta_K(s) = \sum_{I\neq 0}{}' \frac{1}{N_{K/\mathbb{Q}}(I)^s}$$

$$= \sum_{c\in Cl(K)}{}' \zeta_c(s) \qquad \zeta_c(s) = \sum_{[I]=c}{}' \frac{1}{N_{K/\mathbb{Q}}(I)^s}$$

$$\lim_{s\to 1^+} (s-1)\,\zeta_c(s) = \boxed{\frac{2^{r_1}(2\pi)^{r_2}\cdot\mathrm{Reg}_K}{\#\mu_K\cdot\sqrt{|\Delta_K|}}}$$

$c\in Cl(K)$ $\quad I'\subseteq\mathcal{O}_K$ t.q. $[I']=c^{-1}$

$I\subseteq\mathcal{O}_K$ t.q. $[I]=c$ $\quad II'=\alpha\mathcal{O}_K$ para $\alpha\in\mathcal{O}_K\setminus\{0\}$

$$\{I\subseteq\mathcal{O}_K \mid [I]=c\} \longleftrightarrow \{\alpha\mathcal{O}_K \mid \alpha\in I'\}$$

$$N(II') = |N(\alpha)|$$
$$\rotatebox{90}{=}$$
$$N(I)\cdot N(I')$$

$$\zeta_c(s) = \sum_{[I]=c} \frac{1}{N(I)^s} = N(I')^s \sum_{\substack{\alpha\mathcal{O}_K\\ \alpha\in I'}} \frac{1}{|N(\alpha)|^s}$$

$$= N(I')^s\cdot \sum_{\alpha\in I'/\sim}{}' \frac{1}{|N(\alpha)|^s}$$

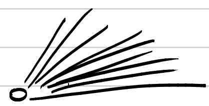
$$K_{\mathbb{C}} = K\otimes_{\mathbb{Q}}\mathbb{C}$$

$$\begin{tikzcd} K \arrow[r, hook, "\Phi"] \arrow[dr] & K_{\mathbb{C}} \\ & K_{\mathbb{R}} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$\mathcal{O}_K^\times \curvearrowright K_{\mathbb{C}}$

$$u\cdot x = \Phi(u)\cdot x$$

$\mathcal{O}_K^\times \curvearrowright K_{\mathbb{R}}^\times \simeq (\mathbb{R}^\times)^n$



Teorema 1 Existe $X\subset K_{\mathbb{R}}^\times$.

1) X es un cono: $x\in X \Rightarrow \forall\lambda>0\;\; \lambda x\in X$

2) X es un dominio fund. resp. $\mathcal{O}_K^\times \curvearrowright K_{\mathbb{R}}^\times$.

$\forall y\in K_{\mathbb{R}}^\times\;\; \exists!\, u\in\mathcal{O}_K^\times\;\; \exists!\, x\in X$ t.q. $y=\Phi(u).x$.

3) $T = \{x \in X \mid \prod_i |x_i| \le 1\}$ es acotado.

$$\mathrm{vol}\, T = \frac{2^{r_1} (2\pi)^{r_2} \cdot \mathrm{Reg}_K}{\# \mu_K}$$

Ejemplo $K = \mathbb{Q}(\sqrt{-3})$ $\quad K_{\mathbb{R}} \simeq \mathbb{R}^2$

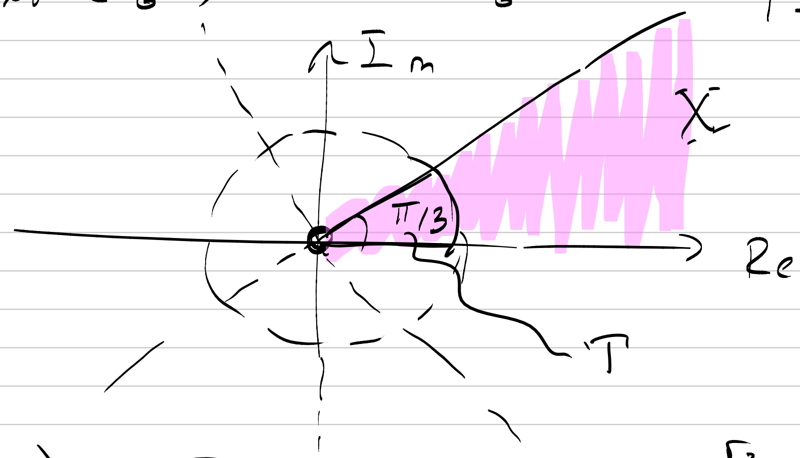
$(x_\sigma, x_{\bar\sigma}) \longmapsto (\mathrm{Re}\, x_\sigma, \mathrm{Im}\, x_\sigma)$

$\mathcal{O}_K^\times = \mu_6(\mathbb{C})$.

$\mathcal{O}_K^\times \curvearrowright K_{\mathbb{R}} \simeq \mathbb{R}^2 \simeq \mathbb{C}$. $\quad$ acción: rotación de $\frac{\pi}{3}$.

$\zeta_6 = \exp\left(\frac{2\pi i}{6}\right)$ $\quad \zeta_6 \cdot z$

$T = \{z \in \mathbb{C}^\times \mid |z| \le 1\}$



$\mathrm{vol}_{Leb.}(T) = \frac{\pi}{6}$. $\quad \mathrm{vol}(T) = 2^{r_2} \cdot \mathrm{vol}_{Leb.}(T) = \frac{\pi}{3}$.

$$\mathrm{vol}(T) = \frac{2^{r_1} (2\pi)^{r_2} \cdot \mathrm{Reg}_K}{\# \mu_K}$$

$= \frac{\pi}{3}$.

·) $\mathrm{Reg}_K = 1$.

·) $\# \mu_K = 6$

·) $r_1 = 0, \quad r_2 = 1$.

Ejemplo $K = \mathbb{Q}(\sqrt{3})$ $\quad K_{\mathbb{R}} \simeq \mathbb{R}^2$

$\mathcal{O}_K^\times = \{\pm u^n \mid n \in \mathbb{Z}\}$ $\quad u = 2 + \sqrt{3}$.

$\simeq \{\pm 1\} \times \langle u \rangle$

$(x, y) \in X \implies x > 0$.

$\Phi(u^n) \cdot (x, y) = ((2+\sqrt{3})^n \cdot x, \; (2-\sqrt{3})^n \cdot y)$.

$e: K_{\mathbb{R}}^\times \longrightarrow \mathbb{R}^2$

$(x, y) \longmapsto (\log|x|, \log|y|)$

$$\mathcal{O}_K^\times \xhookrightarrow{\Phi} K_{\mathbb{R}}^\times \xrightarrow{\ell} \mathbb{R}^2 \supset H = \{(x,y) \mid x+y=0\}.$$

$$\underbrace{\qquad\qquad}_{L_1}$$

$$L(u) = (\log(2+\sqrt{3}),\ \log(2-\sqrt{3})).$$

$$(x,y) \in \mathbb{R}^2 \rightsquigarrow (x,y) = \lambda L(u) + \mu\cdot(1,1).$$

En particular, si $(x,y) \in K_{\mathbb{R}}^\times \Rightarrow$

$$\boxed{\ell(x,y) = \lambda\cdot L(u) + \mu\cdot(1,1)} \qquad \lambda,\ \mu. \qquad (*)$$

Los puntos $(x,y) \in X$ serán
- $x > 0$
- $0 \leq \lambda < 1$.

1) $(x,y) \in K_{\mathbb{R}}^\times \Rightarrow \exists\, v \in \mathcal{O}_K^\times\ \exists (x',y') \in X$ t.q.

$$\Phi(v)\cdot(x,y) = (x',y')$$

$n = \lfloor \lambda \rfloor$ para $(*)$.

$$\ell(\Phi(u^{-n})\cdot(x,y)) = \ell(\Phi(u^{-n})) + \ell(x,y)$$
$$= L(u^{-n}) + \lambda L(u) + \mu\cdot(1,1)$$
$$= \underbrace{(\lambda - n)}_{0 \leq \lambda' < 1}\cdot L(u) + \mu\cdot(1,1).$$

2) $(x,y), (x',y') \in X \iff \begin{cases} \ell(x,y) = \lambda L(u) + \mu\cdot(1,1) \\ \ell(x',y') = \lambda'\cdot L(u) + \mu'(1,1) \end{cases}$

$0 \leq \lambda, \lambda' < 1.$

Supongamos que $\exists\, u^n \in \mathcal{O}_K^\times$ t.q.

$$\Phi(u^n)\cdot(x,y) = (x',y').$$

$$\ell(\Phi(u^n)\cdot(x,y)) = \ell(x',y').$$

$$L_1(u^n) + \ell(x,y) = \ell(x',y').$$

$$(\lambda+n)L_1(u) + \mu\cdot(1,1) = \lambda'\cdot L(u) + \mu'\cdot(1,1)$$

$$\Rightarrow \mu = \mu', \qquad \lambda + n = \lambda' \Rightarrow n = 0.$$

$$\Rightarrow (x,y) = (x',y').$$

$$X = \{ (x,y) \in (\mathbb{R}^\times)^2 \mid \begin{array}{l} x > 0 \\ \log|x| = \lambda \cdot \log(2+\sqrt{3}) + \mu \\ \log|y| = \lambda \cdot \log(2-\sqrt{3}) + \mu \end{array} \}$$

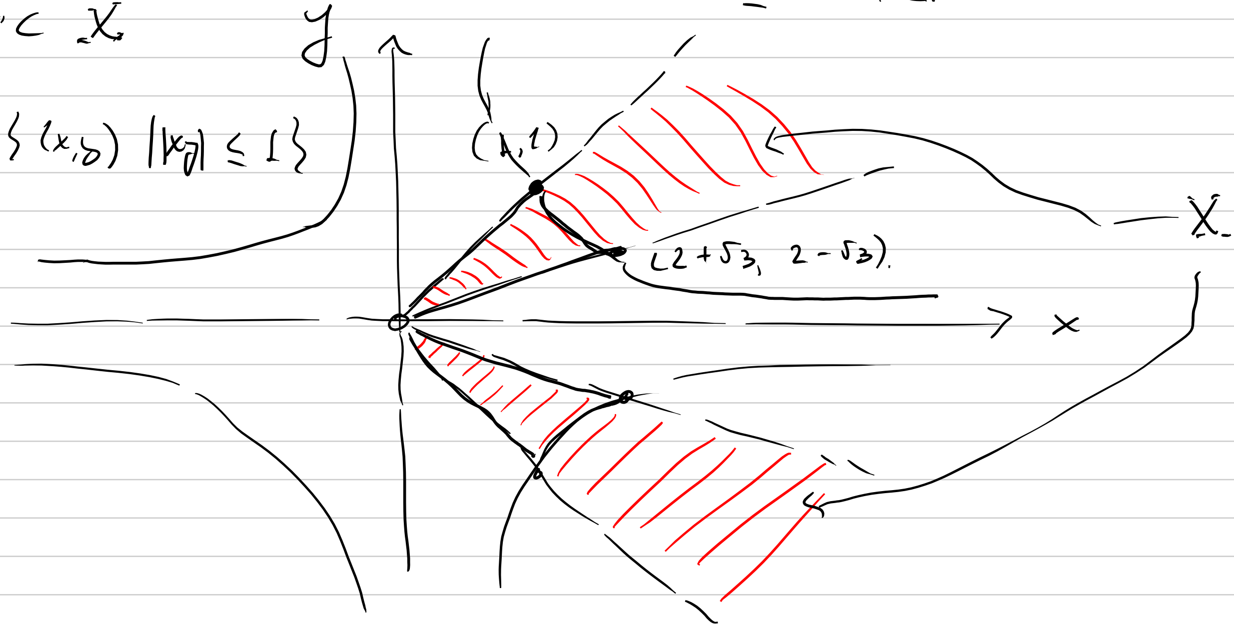
$$0 \le \lambda < 1.$$

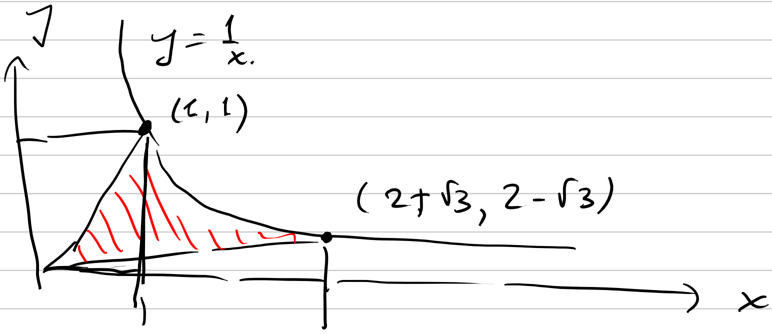
$$K_{\mathbb{R}}^\times \xrightarrow{\ell} \mathbb{R}^2$$

$$\cup$$

$$T \subset X$$

$$T = \{ (x,y) \mid |xy| \le 1 \}$$





$$\mathrm{vol}(T) = 2 \int_1^{2+\sqrt{3}} \frac{dx}{x} = 2 \cdot \log(2+\sqrt{3})$$

$$\mathrm{vol}(T) = \frac{2^{r_1}(2\pi)^{r_2}}{\# \mu_K} \cdot \mathrm{Reg}_K = 2 \cdot \log(2+\sqrt{3})$$

OK $\boxtimes$

$$\zeta_c(s) = N(I')^s \cdot \sum_{\alpha \in I'/\sim} \frac{1}{|N(\alpha)|^s}$$

[X es un dom. fundamental de $\mathcal{O}_K^\times \curvearrowright K_{\mathbb{R}}^\times \Rightarrow$

$\forall\, \alpha \in \mathcal{O}_K \setminus \{0\}\ \exists!\ \beta \in \mathcal{O}_K$ t.q. $\beta \sim \alpha$ y $\Phi(\beta) \in X$.

$\Lambda = \Phi(I') \subset K_{\mathbb{R}}$ $\qquad |N(\alpha)| = \prod_i |\sigma_i(\alpha)|$.

$$\zeta_c(s) = N(I')^s \cdot \sum_{\omega \in \Lambda \cap X} \frac{1}{|N(\omega)|^s}.$$

Teorema 2 Sean X un cono en $(\mathbb{R}^{\times})^n$, $F: X \to \mathbb{R}_{>0}$, $\Lambda \subset \mathbb{R}^n$ retículo de rango completo

1) $\forall x \in X \ \forall \lambda > 0 \quad F(\lambda x) = \lambda^n \cdot F(x)$.
2) $T = \{x \in X \mid F(x) \leq 1\}$ es acotado, de volumen no nulo.

$$Z(s) = \sum_{\omega \in \Lambda \cap X} \frac{1}{F(x)^s} \quad \text{converge para } s > 1.$$

$$\lim_{s \to 1^+} (s-1) Z(s) = \frac{\mathrm{vol}(T)}{\mathrm{covol}(\Lambda)}.$$

Tomamos $X \subset K_{\mathbb{R}}^{\times}$ como antes, $\Lambda = \Phi(I')$, T, $F: x \mapsto |N(x)| = \prod_i |x_i|$.

$$Z(s) = \sum_{\omega \in \Lambda \cap X} \frac{1}{|N(x)|^s}.$$

$$\lim_{s \to 1^+} (s-1) Z(s) = \frac{\mathrm{vol}\ T}{\mathrm{covol}\ \Lambda} = \frac{1}{N(I')} \cdot \frac{2^{r_1} (2\pi)^{r_2} \cdot \mathrm{Reg}_K}{\# \mu_K \sqrt{|\Delta_K|}}$$

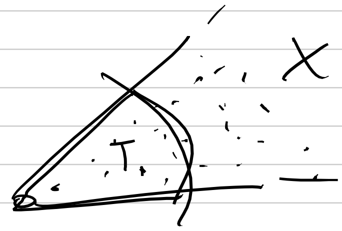
$$\mathrm{covol}\ \Phi(I') = N(I') \cdot \sqrt{|\Delta_K|}$$

$$\lim_{s \to 1^+} (s-1) \zeta_c(s) = N(I') \cdot \lim_{s \to 1^+} (s-1) Z(s).$$

__Conclusión__: $\displaystyle \lim_{s \to 1^+} (s-1) \zeta_c(s) = \frac{2^{r_1} \cdot (2\pi)^{r_2} \mathrm{Reg}_K}{\# \mu_K \cdot \sqrt{|\Delta_K|}}$

Cómo se demuestra el segundo teorema.

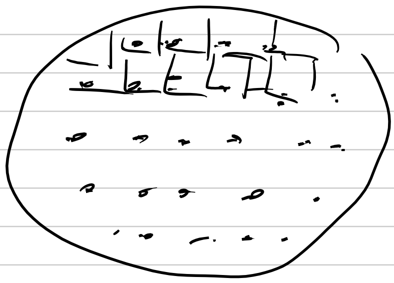
$$T = \{x \in X \mid F(x) \leq 1\}$$



$$Z(s) = \sum_{\omega \in \Lambda \cap X} \frac{1}{F(x)^s}$$

$$\operatorname{vol} T = \lim_{r \to \infty} C(r) \cdot \operatorname{covol}\left(\tfrac{1}{r} \cdot \Lambda\right) = \operatorname{covol} \Lambda \cdot \lim \frac{C(r)}{r^n}$$

$r > 0$



$$C(r) = \# \{\omega \in \tfrac{1}{r} \cdot \Lambda \mid \omega \in T\}$$

$$= \# \{\omega \in \Lambda \mid \omega \in rT\}$$

$$= \# \{\omega \in \Lambda \cap X \mid F(\omega) \leq r^n\}.$$

$F(r\omega) = r^n \cdot F(\omega)$

$\omega \in \Lambda \cap X$ $\qquad \mathbb{Z}^n \subset \mathbb{R}^n$

$$F(\omega_1) \leq F(\omega_2) \leq F(\omega_3) \leq \cdots$$

$$Z(s) = \sum_{k \geq 1} \frac{1}{F(\omega_k)^s}$$

$$\lim_{k \to \infty} \frac{k}{F(\omega_k)} = \frac{\operatorname{vol}(T)}{\operatorname{covol}(\Lambda)}$$

$$\lim_{s \to 1^+} (s-1)\, Z(s) = \frac{\operatorname{vol}(T)}{\operatorname{covol}(\Lambda)}.$$

La próxima sesión:

$K/\mathbb{Q}$ extn abeliana $\rightsquigarrow$ $\zeta_K(s) = \prod_{\chi} L(s,\chi)$.