25/11/2020

$K/\mathbb{Q}$ abeliana $\rightsquigarrow$ $\zeta_K(s) = \prod_{\chi} L(s,\chi)$.

$\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$

$\quad \wr\!\wr$

$\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$

$H \subseteq \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$

$\updownarrow$

$K \subseteq \mathbb{Q}(\zeta_m)$.

Kronecker-Weber: $K/\mathbb{Q}$ extn. abeliano $\Rightarrow$ $K \subseteq \mathbb{Q}(\zeta_m)$

$G \rightsquigarrow \hat{G} = \mathrm{Hom}(G, \mathbb{C}^\times)$

$\widehat{(\mathbb{Z}/m\mathbb{Z})^\times} \simeq \widehat{\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})}$

$\wr\!\wr$ no canónico $\quad \wr\!\wr$

$(\mathbb{Z}/m\mathbb{Z})^\times \simeq \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$

§ Caracteres de gpos abelianos finitos

$G \rightsquigarrow \hat{G}$

$f: G \to H \rightsquigarrow \hat{H} \to \hat{G}$

$(H \xrightarrow{\varphi} \mathbb{C}^\times) \quad (G \xrightarrow{f} H \xrightarrow{\varphi} \mathbb{C}^\times)$

$\widehat{G \times H} \simeq \hat{G} \times \hat{H}$

Lema $G \simeq \hat{G}$ iso no canónico. $\#\hat{G} = \#G$.

Dem $G = C_n$ $\qquad \hat{C}_n = \mathrm{Hom}(C_n, \mathbb{C}^\times) = \mathrm{Hom}(C_n, \mu_n(\mathbb{C}))$

$\simeq \mathrm{Hom}(C_n, C_n) \simeq C_n$.

$G \simeq C_{n_1} \times \cdots \times C_{n_s}$. iso no canónico.

$\hat{G} \simeq \hat{C}_{n_1} \times \cdots \times \hat{C}_{n_s} \simeq C_{n_1} \times \cdots \times C_{n_s} \simeq G$ $\qquad \square$

Proposición (dualidad de Pontryagin)

Sea G gpo abeliano finito.

$\hat{G}$ — gpo de caracteres.

a) iso canónico ev.: $G \simeq \hat{\hat{G}}$ $\quad \hat{G} \to \mathbb{C}^\times$

$g \longmapsto (\chi \longmapsto \chi(g))$

b) apareamiento no degenerado

$$G \times \hat{G} \to \mathbb{C}^\times$$
$$(g, \chi) \longmapsto \chi(g).$$

$\# G = \# \hat{\hat{G}}$

c) $H \subset G \rightsquigarrow H^\perp := \{\chi \in \hat{G} \mid \chi(h) = 1 \ \forall h \in H\}$

$H^\perp \simeq \widehat{G/H} \qquad \hat{H} \simeq \hat{G}/H^\perp.$

d) $(H^\perp)^\perp = H$ (bajo iso $G \simeq \hat{\hat{G}}$).

Comentario 1 | V/k esp. de dim finita.

$V^\vee := \mathrm{Hom}_k(V, k) \qquad V \simeq V^{\vee\vee}$

$v \longmapsto (\varphi \longmapsto \varphi(v))$

Comentario 2 | G - gpo abeliano localmente compacto.

$\hat{G} = \mathrm{Hom}_{\text{Cont.}}(G, \mathbb{T}) \qquad \mathbb{T} = \{z \in \mathbb{C}^\times \mid |z| = 1\}$

a), b), c), d) se cumplen en este caso

§ Caracteres de Dirichlet

$$\chi : (\mathbb{Z}/m\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$$

$m \mid m' \rightsquigarrow \qquad (\mathbb{Z}/m'\mathbb{Z})^\times \longrightarrow (\mathbb{Z}/m\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}$

def El m más pequeño t.q. χ es un carácter de Dirichlet mód m se llama el conductor de χ y se denota por f_χ. Si χ se considera mód f_χ, se dice que χ es primitivo.

Ejemplo $\chi : (\mathbb{Z}/6\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$

$1 \longmapsto +1$

$5 \longmapsto -1.$

$(\mathbb{Z}/6\mathbb{Z})^\times \to (\mathbb{Z}/3\mathbb{Z})^\times \to \mathbb{C}^\times$

$1 \longmapsto 1 \longmapsto +1$

$5 \longmapsto 2 \longmapsto -1$

$f_\chi = 3.$

(f_χ "f" Führer)

Def $\chi : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^* \rightsquigarrow \chi : \mathbb{Z} \to \mathbb{C}$

$\chi(n) = 0$ si $\operatorname{mcd}(n, f_\chi) \neq 1$.

Si χ, χ' son primitivos,

$$\chi\chi' : (\mathbb{Z}/\operatorname{mcm}(f_\chi, f_{\chi'})\mathbb{Z})^* \to \mathbb{C}^*$$

si $\operatorname{mcd}(f_\chi, f_{\chi'}) = 1 \Rightarrow f_{\chi\chi'} = f_\chi \cdot f_{\chi'}$.

§ **Caracteres de Dirichlet y subcampos de $\mathbb{Q}(\zeta_m)$**

$$\chi : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^* \rightsquigarrow \chi : \overbrace{\operatorname{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})}^{G} \to \mathbb{C}^*.$$

$$H = \ker \chi \longrightarrow K = \mathbb{Q}(\zeta_m)^H$$

$$\chi : G \longrightarrow \mathbb{C}^*$$
$$G \to G/H \to \mathbb{C}^*$$
$$G/H = \operatorname{Gal}(K/\mathbb{Q}).$$

•) En gral, sea X un gpo finito de caracteres de Dirichlet. $m = \operatorname{mcm}(f_\chi \mid \chi \in X)$

$$G = \operatorname{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}).$$

$$H = \bigcap_{\chi \in X} \ker \chi \longrightarrow K = \mathbb{Q}(\zeta_m)^H$$

$$\begin{array}{ccc} G & \longrightarrow & \mathbb{C}^* \\ & \searrow & \uparrow \exists! \\ & & \operatorname{Gal}(K/\mathbb{Q}) \end{array}$$

Cada χ es un carácter de Galois de $K/\mathbb{Q}$.

$$X \longleftrightarrow \widehat{\operatorname{Gal}(K/\mathbb{Q})}$$

$$\operatorname{Gal}(K/\mathbb{Q}) \times X \longrightarrow \mathbb{C}^*$$

•) $F \subseteq K \rightsquigarrow Y = \operatorname{Gal}(K/F)^{\perp} \simeq \widehat{\left(\dfrac{\operatorname{Gal}(K/\mathbb{Q})}{\operatorname{Gal}(K/F)}\right)}$

$$\simeq \widehat{\operatorname{Gal}(F/\mathbb{Q})}$$

⇒) Viceversa, $Y \subseteq X \rightsquigarrow K^{(Y^\perp)}$ $\quad Y^\perp \simeq \mathrm{Gal}(K/F)$.

$$Y \cong (Y^\perp)^\perp \simeq \mathrm{Gal}(K/F)^\perp \simeq \widehat{\mathrm{Gal}(F/\mathbb{Q})}.$$

Teorema $X \rightsquigarrow K \subseteq \mathbb{Q}(\zeta_m)$

$$\{\text{Subgrupos } Y \subseteq X\} \underset{\mathrm{Gal}(K/F)^\perp \,\leftarrow\!\shortmid\, F}{\overset{Y \mapsto K^{(Y^\perp)}}{\rightleftarrows}} \{\text{subcampos } F \subseteq K\}$$

a) $F_1 \subseteq F_2 \Leftrightarrow Y_1 \subseteq Y_2$

b) $\langle Y_1, Y_2 \rangle \longleftrightarrow F_1 F_2$

En particular,

$$\{\text{subgrupos } X \subseteq \widehat{(\mathbb{Z}/m\mathbb{Z})^\times}\} \longleftrightarrow \{\text{subcampos } K \subseteq \mathbb{Q}(\zeta_m)\}.$$

(Kronecker-Weber) todo $K/\mathbb{Q}$ abeliano es un subcampo de algún $\mathbb{Q}(\zeta_m)$.

$K/\mathbb{Q}$ abeliano $\rightsquigarrow X$ gpo de caracteres de Dirichlet.

§ Caracteres de Dirichlet y ramificación

Teorema Sean X un gpo de caracteres de Dirichlet y $K \subseteq \mathbb{Q}(\zeta_m)$ subcampo correspondiente.

$p \in \mathbb{Z}$ no se ramifica en $K/\mathbb{Q}$ $\Leftrightarrow$ $\chi(p) \neq 0$ $\forall \chi \in X$.

Teorema X – gpo de caracteres de Dirichlet
$K \subseteq \mathbb{Q}(\zeta_m)$ ~ campo correspondiente
$p \in \mathbb{Z}$ – primo fijo.

$$Y = \{\chi \in X \mid \chi(p) \neq 0\}$$
$$Z = \{\chi \in X \mid \chi(p) = 1\}$$

Y/Z es cíclico, $\quad [X:Y] = e_p$ ✓ gpo de inercia
$[Y:Z] = f_p$ ~ gpo de descomp.
$\# Z = g_p$

$$\begin{tikzcd}
 & \mathbb{Q}(\zeta_m) \arrow[d, dash] & \\
X \supset Y \arrow[r, squiggly] & K \supset \mathcal{O}_K \arrow[d, dash] & \\
 & \mathbb{Q} \supset \mathbb{Z} \ni p &
\end{tikzcd}$$

$$p\mathcal{O}_K = (\mathfrak{p}_1 \cdots \mathfrak{p}_{g_p})^{e_p}$$

$$f_p = [\mathcal{O}_K/\mathfrak{p}_i : \mathbb{F}_p].$$

Idea principal: $F = K^{Y^\perp}$ ~ subextn. más grande donde p no se ramifica.

$K^{I(\mathfrak{p}|p)}$ ~ campo de inercia.

$$\Rightarrow F = K^{\mathrm{Gal}(K/F)} = K^{I(\mathfrak{p}|p)} \Rightarrow \mathrm{Gal}(K/F) \cong I(\mathfrak{p}|p)$$

$$\# I(\mathfrak{p}|p) = e_p = \# \mathrm{Gal}(K/F) = [X:Y].$$

$$\begin{tikzcd}
K \arrow[d, dash, "e_p"] \\
K^D \arrow[d, dash, "f_p"] \\
K^I \arrow[d, dash, "g_p"] \\
\mathbb{Q}
\end{tikzcd}$$

Teorema Sean X un gpo de caracteres de Dirichlet, $K \in \mathbb{Q}(\zeta_m)$ - campo correspondiente.

$$\zeta_K(s) = \prod_{\chi \in X} L(s, \chi)$$

Dem.

$$\zeta_K(s) = \prod_p \prod_{\mathfrak{p} | p} \frac{1}{1 - N(\mathfrak{p})^{-s}}$$

$$p\mathcal{O}_K = (\mathfrak{p}_1 \cdots \mathfrak{p}_g)^e \qquad N(\mathfrak{p}_i) = p^f$$

$$\rightsquigarrow \text{factor} \prod_{1 \le i \le g} \frac{1}{1 - N(\mathfrak{p}_i)^{-s}} = \frac{1}{(1 - p^{-fs})^g}$$

$$\zeta_K(s) = \prod_p \frac{1}{(1 - p^{-f_p \cdot s})^{g_p}}$$

$$\prod_{\chi \in X} L(s, \chi) = \prod_p \prod_{\chi \in X} \frac{1}{1 - \chi(p) \cdot p^{-s}}.$$

$$\prod_{\chi \in X} \frac{1}{1 - \chi(p) \cdot p^{-s}} = \prod_{\chi \in Y} \frac{1}{1 - \chi(p) \cdot p^{-s}}$$

$$Y = \{\chi \in X \mid \chi(p) \neq 0\}$$
$$Z = \{\chi \in X \mid \chi(p) = 1\}$$

Y/Z cíclico de orden f_p. $\#Z = g_p$.

$$\prod_{0 \le k \le f-1} \frac{1}{1 - \zeta_f^k p^{-s}} = \frac{1}{1 - p^{-fs}}$$

$$\prod_{x \in Y} \frac{1}{1 - x(p) \cdot p^{-s}} = \frac{1}{\left(1 - p^{-f_p s}\right)^{g_p}}.$$

$\square$

Corolario Si $x \neq 1 \Rightarrow L(1, x) \neq 0$.

Dem. $X = \langle x \rangle$

$$\zeta_K(s) = \prod_{x \in X} L(s, x)$$

$$= \zeta(s) \cdot \prod_{k \neq 0} L(s, x^k)$$

$\zeta_K(s)$ tiene polo simple en $s = 1$.

$\zeta(s)$ tb. tiene polo simple en $s = 1$.

$\Rightarrow L(s, x)$ no puede tener cero en $s = 1$ $\square$

Ejemplo

$$\begin{array}{c} \mathbb{Q}(\zeta_7) \\ \Big|\ 2 \\ K = \mathbb{Q}(\zeta_7 + \zeta_7^{-1}) \\ \Big|\ 3 \\ \mathbb{Q} \end{array} \quad (\mathbb{Z}/7\mathbb{Z})^\times$$

conjugación compleja

índice 3

$K \longleftrightarrow H \subset \mathrm{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q})$

$\longleftrightarrow \underbrace{X}_{\text{orden } 3} \subset \widehat{(\mathbb{Z}/7\mathbb{Z})^\times}$

K debe corresponder a un subgrupo

$X \underset{2}{\subset} \widehat{(\mathbb{Z}/7\mathbb{Z})^\times}$

$X = \langle x \rangle$, donde x es cúbico mód 7.

$$\chi: \quad \begin{aligned} 1 &\longmapsto 1 \\ 2 &\longmapsto \zeta_3^2 \\ 3 &\longmapsto \zeta_3 \\ 4 &\longmapsto \zeta_3 \\ 5 &\longmapsto \zeta_3^2 \\ 6 &\longmapsto 1 \end{aligned}$$

$(\mathbb{Z}/7\mathbb{Z})^\times$

$X = \{1, \chi, \chi^2\}$ $\qquad \chi^2 = \overline{\chi}$.

$\zeta_K(s) = \zeta(s)\, L(1,\chi)\, L(1,\chi^2)$

<u>Comentario</u> Para <u>cualquier</u> campo de # $K/\mathbb{Q}$ (no necesariamente abeliano)

$$\zeta_K(s) = \prod_{\rho} \boxed{L(s,\rho)}$$

series de Artin

$\rho: \mathrm{Gal}(K/\mathbb{Q}) \longrightarrow GL_n(\mathbb{C})$
representaciones irreducible

Si $\mathrm{Gal}(K/\mathbb{Q})$ abeliano $\Longrightarrow$
las repr. irreducibles son caracteres.

$\chi: \mathrm{Gal}(K/\mathbb{Q}) \longrightarrow GL_1(\mathbb{C}) = \mathbb{C}^\times$