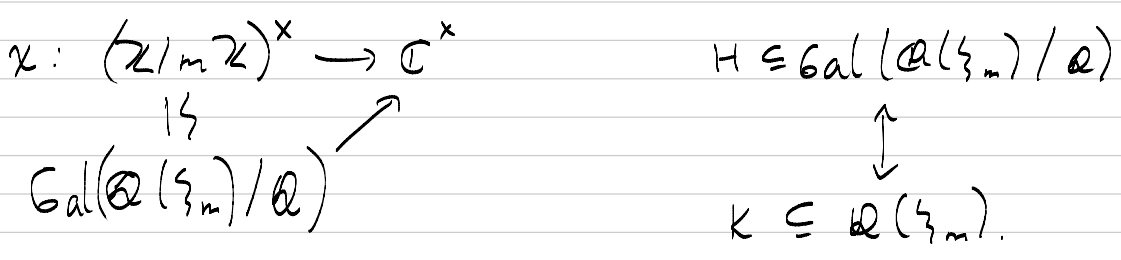


K/\mathbb{Q} abeliana $\implies \chi_K(s) = \prod_x L(s, \chi)$



Kronecker-Weber! K/\mathbb{Q} extn. abeliana $\implies K \subseteq \mathbb{Q}(\zeta_m)$

$G \cong \hat{G} = \text{Hom}(G, \mathbb{C}^\times)$
 $(\mathbb{Z}/m\mathbb{Z})^\times \cong \text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$
 $\cong \text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ (no canónico)

Caracteres de grupos abelianos finitos

$G \cong \hat{G}$
 $f: G \rightarrow H \implies \hat{H} \rightarrow \hat{G}$
 $(H \xrightarrow{\varphi} \mathbb{C}^\times) \quad (G \xrightarrow{f} H \xrightarrow{\varphi} \mathbb{C}^\times)$
 $\hat{G} \times \hat{H} \cong \widehat{G \times H}$

Lema $G \cong \hat{G}$ iso no canónico. $\# \hat{G} = \# G$.

Dem $G = C_n \implies \hat{C}_n = \text{Hom}(C_n, \mathbb{C}^\times) = \text{Hom}(C_n, \mu_n(\mathbb{C}))$
 $\cong \text{Hom}(C_n, C_n) \cong C_n$

$G \cong C_{n_1} \times \dots \times C_{n_s}$ iso no canónico.

$\hat{G} \cong \hat{C}_{n_1} \times \dots \times \hat{C}_{n_s} \cong C_{n_1} \times \dots \times C_{n_s} \cong G$

Proposición (dualidad de Pontryagin)

Sea G grupo abeliano finito.
 \hat{G} - grupo de caracteres.

a) iso canónico ev.: $G \cong \widehat{\widehat{G}} \quad \widehat{G} \rightarrow \mathbb{C}^\times$
 $g \mapsto (x \mapsto x(g))$

b) apareamiento no degenerado
 $G \times \widehat{G} \rightarrow \mathbb{C}^\times$
 $(g, \chi) \mapsto \chi(g)$
 $\#G = \#\widehat{\widehat{G}}$

c) $H \subset G \rightsquigarrow H^\perp := \{ \chi \in \widehat{G} \mid \chi(h) = 1 \ \forall h \in H \}$
 $H^\perp \cong \widehat{G/H} \quad \widehat{H} \cong \widehat{G}/H^\perp$

d) $(H^\perp)^\perp = H$ (bajo iso $G \cong \widehat{\widehat{G}}$)

Comentario 1 V/k esp. de dim finita.

$V^\vee := \text{Hom}_{\mathbb{k}}(V, \mathbb{k}) \quad V \cong V^{\vee\vee}$
 $\sigma \mapsto (\varphi \mapsto \varphi(\sigma))$

Comentario 2

G -gpo abeliano localmente compacto.

$\widehat{G} = \text{Hom}_{\text{Cont.}}(G, \mathbb{T}) \quad \mathbb{T} = \{ z \in \mathbb{C}^\times \mid |z| = 1 \}$

a), b), c), d) se cumplen en este caso

Caracteres de Dirichlet

$\chi: (\mathbb{Z}/m\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$

$m \mid m' \rightsquigarrow (\mathbb{Z}/m'\mathbb{Z})^\times \rightarrow (\mathbb{Z}/m\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}$

def El m más pequeño t.q. χ es un carácter de Dirichlet mód m se llama

el conductor de χ y se denota

por f_χ . Si χ se considere mód f_χ ,

se dice que χ es primitivo.

Ejemplo $\chi: (\mathbb{Z}/6\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$

$1 \mapsto +1$
 $5 \mapsto -1$

$f_\chi = 3$.

$(\mathbb{Z}/6\mathbb{Z})^\times \rightarrow (\mathbb{Z}/3\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$

$1 \mapsto 1 \mapsto +1$
 $5 \mapsto 2 \mapsto -1$

(f_x "f" Führer.)

Def $\chi: (\mathbb{Z}/m\mathbb{Z})^\times \rightarrow \mathbb{C}^\times \rightsquigarrow \chi: \mathbb{Z} \rightarrow \mathbb{C}$
 $\chi(n) = 0$ si $\gcd(n, f_x) \neq 1$.

Si χ, χ' son primitivos,

$$\chi\chi': (\mathbb{Z}/\text{mcm}(f_x, f_{\chi'})\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$$

si $\gcd(f_x, f_{\chi'}) = 1 \Rightarrow f_{\chi\chi'} = f_x f_{\chi'}$.

Caracteres de Dirichlet y subgrupos de $\mathbb{Q}(\zeta_m)$

$$\chi: (\mathbb{Z}/m\mathbb{Z})^\times \rightarrow \mathbb{C}^\times \rightsquigarrow \chi: \text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \rightarrow \mathbb{C}^\times$$

$$H = \ker \chi \rightsquigarrow K = \mathbb{Q}(\zeta_m)^H$$

$$\begin{array}{ccc} \chi: & G & \longrightarrow \mathbb{C}^\times \\ & \searrow & \nearrow \\ & G/H & \\ & \cong & \text{Gal}(K/\mathbb{Q}) \end{array}$$

En gen, sea X_- un grp finito de caracteres de Dirichlet.
 $m = \text{mcm}(f_x \mid x \in X_-)$

$$G = \text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$$

$$H = \bigcap_{\chi \in X_-} \ker \chi \rightsquigarrow K = \mathbb{Q}(\zeta_m)^H$$

$$\begin{array}{ccc} \chi: & G & \longrightarrow \mathbb{C}^\times \\ & \searrow & \nearrow \\ & \text{Gal}(K/\mathbb{Q}) & \end{array}$$

Cada χ es un carácter de Galois de K/\mathbb{Q} .

$$X_- \longleftrightarrow \widehat{\text{Gal}(K/\mathbb{Q})}$$

$$\text{Gal}(K/\mathbb{Q}) \times X_- \longrightarrow \mathbb{C}^\times$$

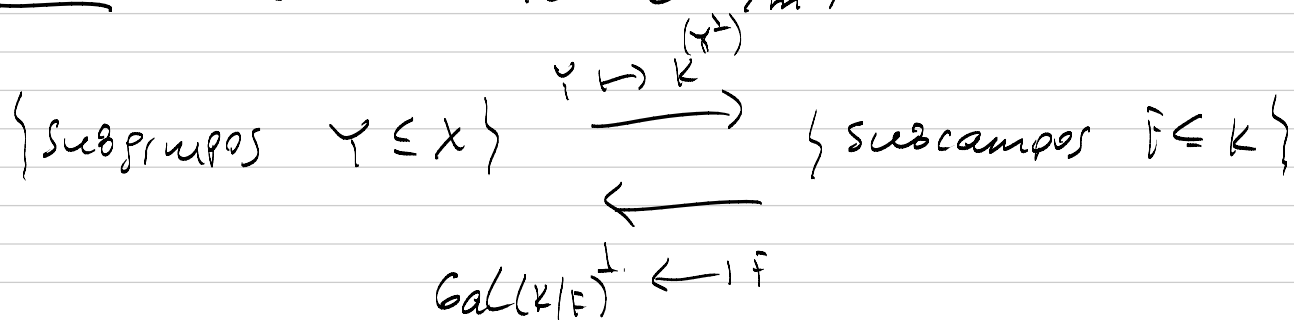
$F \subseteq K \rightsquigarrow Y = \text{Gal}(K/F)^\perp \simeq \widehat{\text{Gal}(F/\mathbb{Q})}$

$$\simeq \widehat{\text{Gal}(K/\mathbb{Q}) / \text{Gal}(K/F)}$$

7) Vicerusa, $Y \subseteq X \rightsquigarrow K^{(Y^\perp)}$ $Y^\perp \cong \text{Gal}(K/F)$.

$$Y \cong (Y^\perp)^\perp \cong \text{Gal}(K/F)^\perp \cong \text{Gal}(F/\mathbb{Q})$$

Teorema $X_i \rightsquigarrow K \subseteq \mathbb{Q}(\zeta_m)$



a) $F_1 \subseteq F_2 \iff Y_1 \subseteq Y_2$

b) $\langle Y_1, Y_2 \rangle \iff F_1 F_2$

En particular,

$$\{ \text{subgrupos } X \subseteq (\mathbb{Z}/m\mathbb{Z})^\times \} \longleftrightarrow \{ \text{subcampos } K \subseteq \mathbb{Q}(\zeta_m) \}$$

Kronecker-Weber todo K/\mathbb{Q} abeliano es un subcampo de algún $\mathbb{Q}(\zeta_m)$.

K/\mathbb{Q} abeliano $\rightsquigarrow X$ grp de caracteres de Dirichlet.

Caracteres de Dirichlet y ramificación

Teorema Sean X un grp de caracteres de Dirichlet y $K \subseteq \mathbb{Q}(\zeta_m)$ subcampo correspondiente.

$p \in \mathbb{Z}$ no se ramifica en $K \iff \chi(p) \neq 0 \quad \forall \chi \in X$

Teorema

X - gpo de caracteres de Dirichlet,
 $K \subseteq \mathbb{Q}(\zeta_m) \sim$ campo correspondiente
 $p \in \mathbb{Z}$ - primo distinto.

$$Y = \{ \chi \in X \mid \chi(p) \neq 0 \}$$

$$Z = \{ \chi \in X \mid \chi(p) = 1 \}$$

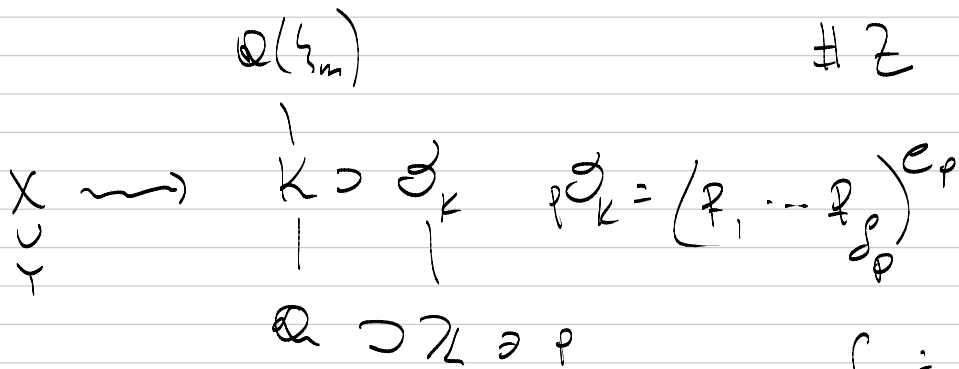
Y/Z es cíclico,

$$[X:Y] = e_p$$

$$[Y:Z] = f_p$$

$$\# Z = g_p$$

gpo de inercia
gpo de descomp.



$$f_p = [\mathcal{O}_K / \mathfrak{p}_i : \mathbb{F}_p]$$

(Idea principal): $F = K^Y \sim$ subextn. más grande donde p no se ramifica.

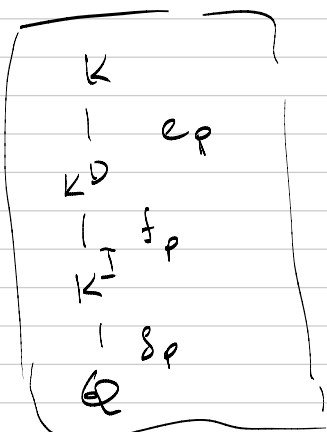
$K \sim$ campo de inercia.

$$\Rightarrow F=K = K \stackrel{\text{Gal}(K/F)}{\cong} I(\mathfrak{p}|p)$$

$$\Rightarrow \text{Gal}(K/F) \cong I(\mathfrak{p}|p)$$

$$\# I(\mathfrak{p}|p) = e_p = \# \text{Gal}(K/F)$$

$$\parallel [X:Y]$$



Teorema

Sean X_i un gpo de caracteres de Dirichlet, $K \in \mathbb{Q}(\zeta_m)$ - campo correspondiente.

$$\zeta_K(s) = \prod_{\chi \in X_i} L(s, \chi)$$

Dem.

$$\zeta_K(s) = \prod_p \prod_{\chi \neq 1} \frac{1}{1 - \chi(p) \cdot p^{-s}}$$

$$p \mathfrak{g}_K = (p_1 \cdots p_g)^e \quad \nu(p_i) = p^f$$

$$\leadsto \text{factor } \prod_{1 \leq i \leq g} \frac{1}{1 - \chi(p_i) \cdot p^{-s}} = \frac{1}{(1 - p^{-fs})^g}$$

$$\zeta_K(s) = \prod_p \frac{1}{(1 - p^{-fs})^g}$$

$$\prod_{\chi \in X_i} L(s, \chi) = \prod_p \prod_{\chi \in X} \frac{1}{1 - \chi(p) \cdot p^{-s}}$$

$$\prod_{\chi \in X} \frac{1}{1 - \chi(p) \cdot p^{-s}} = \prod_{\chi \in Y} \frac{1}{1 - \chi(p) \cdot p^{-s}}$$

$$Y = \{ \chi \in X \mid \chi(p) \neq 0 \}$$

$$Z = \{ \chi \in X \mid \chi(p) = 1 \}$$

Y/Z cíclico de orden f . $\# Z = g$.

$$\prod_{0 \leq k \leq f-1} \frac{1}{1 - \zeta_f^k p^{-s}} = \frac{1}{1 - p^{-fs}}$$

$$\prod_{\chi \in \mathcal{X}} \frac{1}{1 - \chi(p) \cdot p^{-s}} = \frac{1}{(1 - p^{-s})^g} \quad \square$$

Corolario Si $\chi \neq 1 \Rightarrow L(s, \chi) \neq 0$.

Dem. $\mathcal{X} = \langle \chi \rangle$

$$\begin{aligned} \zeta_k(s) &= \prod_{\chi \in \mathcal{X}} L(s, \chi) \\ &= \zeta(s) \cdot \prod_{\chi \neq 1} L(s, \chi^k) \end{aligned}$$

$\zeta_k(s)$ tiene polo simple en $s=1$.

$\zeta(s)$ tb. tiene polo simple en $s=1$.

$\Rightarrow L(s, \chi)$ no puede tener cero en $s=1$ □

Ejemplo $\mathbb{Q}(\zeta_7)$

$$\begin{array}{c} \mathbb{Q}(\zeta_7) \\ | \quad 2 \\ \mathbb{K} = \mathbb{Q}(\zeta_7 + \zeta_7^{-1}) \\ | \quad 3 \\ \mathbb{Q} \end{array}$$

$$(\mathbb{Z}/7\mathbb{Z})^\times$$

conjugación compleja
índice 3

$$\mathbb{K} \leftrightarrow H \subset \text{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q})$$

$$\leftrightarrow X \subset (\mathbb{Z}/7\mathbb{Z})^\times$$

orden 3.

\mathbb{K} debe corresponder a un subgrupo

$$X \subset (\mathbb{Z}/7\mathbb{Z})^\times$$

$X = \langle \chi \rangle$, donde χ es cúbico módulo 7.

$$\chi: \begin{array}{l} 1 \mapsto 1 \\ 2 \mapsto \zeta_3^2 \\ 3 \mapsto \zeta_3 \\ 4 \mapsto \zeta_3^2 \\ 5 \mapsto \zeta_3 \\ 6 \mapsto 1 \end{array}$$

$$X = \{1, x, x^2\} \quad x^2 = \overline{x}$$

$$\zeta_k(s) = \zeta(s) \cdot L(s, \chi) \cdot L(s, \chi^2)$$

$(\mathbb{Z}/7\mathbb{Z})^\times$

Comentario

Para cualquier campo
 $k \neq \mathbb{Q}$ (no necesariamente
 abeliano)

$$\zeta_k(s) = \prod_p L(s, \rho)$$

series de Artin

$\rho: \text{Gal}(k/\mathbb{Q}) \rightarrow \text{GL}_n(\mathbb{C})$
 representaciones irreducibles

Si $\text{Gal}(k/\mathbb{Q})$ abeliano \implies
 las repr. irreducibles son caracteres.

$$\chi: \text{Gal}(k/\mathbb{Q}) \rightarrow \text{GL}_1(\mathbb{C}) = \mathbb{C}^\times$$