# Zeta-values of one-dimensional arithmetic schemes at $n < 0$

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**Abstract**

Let $X \to \operatorname{Spec} \mathbb{Z}$ be an arithmetic scheme (separated, of finite type) of Krull dimension 1. We write down a formula for the special value of $\zeta(X, s)$ at $s = n < 0$, in terms of étale motivic cohomology of $X$ and a regulator. We prove it in the case when for each generic point $\eta \in X$ with $\operatorname{char} \kappa(\eta) = 0$, the extension $\kappa(\eta)/\mathbb{Q}$ is abelian. Further, we conjecture that the formula holds for any one-dimensional arithmetic scheme.

This is a consequence of Weil-étale formalism developed by the author in [Bes2020, Bes2021], following the work of Flach and Morin [FM2018]. In particular, we calculate Weil-étale cohomology of one-dimensional arithmetic schemes.

## Contents



## 1 Introduction

Let $X$ be an **arithmetic scheme**, by which we will mean throughout this text that it is separated and of finite type over $\operatorname{Spec} \mathbb{Z}$. The **zeta function** associated to $X$ (see e.g. [Ser1965]) is given by

$$\zeta(X, s) = \prod_{\substack{x \in X \\ \text{closed pt.}}} \frac{1}{1 - \#\kappa(x)^{-s}},$$

where $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ denotes the residue field of a point. The above product converges for $\operatorname{Re} s > \dim X$, and conjecturally admits a meromorphic continuation to the whole complex plane. Even though this is a wide open conjecture in general, this is well-known for $\dim X = 1$, which will be the case of interest for us.

If $\zeta(X, s)$ admits a meromorphic continuation around $s = n$, then denote by

$$d_n = \operatorname{ord}_{s=n} \zeta(X, s) \tag{1.1}$$

the vanishing order of $\zeta(X,s)$ at $s=n$. The corresponding special value of $\zeta(X,s)$ at $s=n$ is defined to be the leading nonzero coefficient of the Taylor expansion:

$$\zeta^*(X,n) = \lim_{s\to n} (s-n)^{-d_n}\,\zeta(X,s).$$

A primordial example of special value formulas is Dirichlet's **analytic class number formula**. Namely, for a number field $F/\mathbb{Q}$, we will denote by $\mathcal{O}_F$ the corresponding ring of integers. Then $\zeta_F(s) = \zeta(X,s)$ for $X = \operatorname{Spec}\mathcal{O}_F$ is the Dedekind zeta function attached to $F$. It is an easy consequence of the well-known functional equation for $\zeta_F(s)$ that it has a zero at $s=0$ of order $r_1+r_2-1$, where $r_1$ (resp. $r_2$) is the number of real embeddings $F\hookrightarrow\mathbb{R}$ (resp. conjugate pairs of complex embeddings $F\hookrightarrow\mathbb{C}$). The corresponding special value is given by

$$\zeta_F^*(0) = -\frac{h_F}{\omega_F}\,R_F, \tag{1.2}$$

where $h_F = \#\operatorname{Pic}(\mathcal{O}_F)$ is the class number, $\omega_F = \#(\mathcal{O}_F)^\times_{tors}$ is the number of roots of unity in $F$, and $R_F$ is the regulator, which is a positive real number; see for instance [Neu1999, §VII.5].

It is natural to ask whether such formulas exist for $s=n\in\mathbb{Z}$ other than $s=0$ (or $s=1$, which is related via the functional equation to $s=0$). For this one has to find a suitable generalization for the numbers $h_F$ and $\omega_F$ and the regulator $R_F$. Many special value conjectures, of varying generality, originate from this question.

Lichtenbaum proposed in his pioneering work [Lic1973] formulas in terms of algebraic $K$-theory. Later this was also restated in terms of cohomology groups $H^i(\operatorname{Spec}\mathcal{O}_F[1/p]_{ét},\mathbb{Z}_p(n))$ for $i=1,2$ and all primes $p$, and the corresponding formula is known as "cohomological Lichtenbaum conjecture"; see for instance [HK2003]. We will not go into the details, since instead of working with $p$-adic cohomology $H^i(\operatorname{Spec}\mathcal{O}_F[1/p]_{ét},\mathbb{Z}_p(n))$ for varying $p$, it will more convenient for us to use motivic cohomology.

A suitable generalization of $R_F$ are the "higher regulators" that have been considered starting from Borel's work [Bor1977], and later on by Beilinson [Bei1984].

We will not make any attempt at giving a proper historical overview of the subject and writing down all conjectural formulas; the interested reader may consult lecture notes [Kol2004] and surveys [Gon2005, Kah2005].

Lichtenbaum has envisioned a more recent research program, known as **Weil-étale cohomology**; see e.g. [Lic2005, Lic2009b, Lic2009a]. It suggests that for an arithmetic scheme $X$ the special value of $\zeta(X,s)$ at $s=n\in\mathbb{Z}$ should be expressible in terms of Weil-étale cohomology $H^i_{W,c}(X,\mathbb{Z}(n))$, which is a suitable modification of the (étale) motivic cohomology of $X$, that gives finitely generated abelian groups. Flach and Morin gave in [FM2018] an explicit construction of Weil-étale cohomology for a proper and regular arithmetic scheme $X$, and stated a precise conjectural relation of $H^i_{W,c}(X,\mathbb{Z}(n))$ to $\zeta^*(X,n)$.

In particular, in [FM2018, §5.8.3] they write down explicitly their special value formula for the case of $X = \operatorname{Spec}\mathcal{O}_F$. For us it will be convenient to put it as

$$\zeta_F^*(n) = \pm\frac{|H^0(X_{ét},\mathbb{Z}^c(n))|}{|H^{-1}(X_{ét},\mathbb{Z}^c(n))_{tors}|}\,R_{F,n} \quad \text{for } n\le 0. \tag{1.3}$$

The definition of groups $H^i(X_{ét},\mathbb{Z}^c(n))$ is reviewed below, and the regulator $R_{F,n} = R_{\operatorname{Spec}\mathcal{O}_F,n}$ is defined in §5. According to [FM2018, Proposition 5.35], the above formula holds unconditionally for abelian number fields $F/\mathbb{Q}$, via reduction to the Tamagawa number conjecture of Bloch–Kato–Fontaine–Perrin-Riou.

In particular, if we take $n=0$, then $\mathbb{Z}^c(0)\cong\mathbb{G}_m[1]$, and $R_{F,0}$ is the usual regulator of Dirichlet, so that (1.3) becomes

$$\zeta_F^*(0) = \pm\frac{|H^1(\operatorname{Spec}\mathcal{O}_{F,ét},\mathbb{G}_m)|}{|H^0(\operatorname{Spec}\mathcal{O}_{F,ét},\mathbb{G}_m)_{tors}|}\,R_F = \pm\frac{|\operatorname{Pic}(\mathcal{O}_F)|}{|(\mathcal{O}_F)^\times_{tors}|}\,R_F,$$

which is the classical formula (1.2).

In case of $n < 0$, the author has extended in [Bes2020, Bes2021] the work of Flach and Morin to any arithmetic scheme $X$ (thus removing the assumption that $X$ is proper or regular). In this text we would like to work out explicitly the corresponding special value formula for one-dimensional arithmetic schemes.

In order to state an unconditional result, it will be convenient to introduce the following definition.

**1.1. Definition.** We say that a one-dimensional arithmetic scheme $X$ is **abelian** if each generic point $\eta \in X$ with $\operatorname{char} \kappa(\eta) = 0$ corresponds to an abelian extension $\kappa(\eta)/\mathbb{Q}$.

In particular, if $X$ lives entirely in positive characteristic, then it is abelian. The goal is to prove the following result.

**1.2. Theorem.** *For an abelian one-dimensional arithmetic scheme $X$, the special value of $\zeta(X, s)$ at $s = n < 0$ is given by*

$$\zeta^*(X, n) = \pm 2^\delta \, \frac{|H^0(X_{\acute{e}t}, \mathbb{Z}^c(n)|}{|H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{tors}| \cdot |H^1(X_{\acute{e}t}, \mathbb{Z}^c(n))|} \, R_{X,n}. \tag{1.4}$$

*Here*

- *$H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ is a version of (étale) motivic cohomology, reviewed below;*
- *the "correcting factor" $2^\delta$ is given by*

$$\delta = \delta_{X,n} = \begin{cases} r_1, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \tag{1.5}$$

  *and $r_1 = |X(\mathbb{R})|$ is the number of real places of $X$,*
- *$R_{X,n}$ is a positive real number, defined in §5 via Beilinson's regulator map.*

We further conjecture that the special value formula still holds for non-abelian one-dimensional schemes. This is equivalent to Tamagawa number conjecture for non-abelian number fields.

We will give two proofs of (1.4). The first proof can be found in §6 after some preliminary calculations of motivic and Weil-étale cohomology. We observe that the special value formula is the same as conjecture $\mathbf{C}(X, n)$ that was stated in [Bes2021], specialized to $\dim X = 1$ and written out more explicitly. The second proof in §7 is more elementary and self-contained. Essentially it is the same argument, only written out in a more explicit manner.

The purpose of this text is twofold. Firstly, we establish a new special value formula, generalizing several formulas that can be found in the literature. Secondly, we go through the construction of Weil-étale cohomology $H^i_{W,c}(X, \mathbb{Z}(n))$ from [Bes2020], explaining it in the case of one-dimensional schemes. It is not very surprising that such a special value formula exists, but the right cohomological invariants to state it are suggested by the Weil-étale framework.

This text was inspired by the work of Jordan and Poonen [JP2020], where the authors write down a formula for $\zeta^*(X, 1)$, where $X$ is an affine reduced one-dimensional arithmetic scheme. The affine and reduced restriction does not appear in our case, since we work with different invariants. In particular, as $\zeta(X, s) = \zeta(X_{red}, s)$, the "right" invariants should not distinguish between $X$ and $X_{red}$, and motivic cohomology satisfies this property.

## Notation and conventions

Throughout this text $X$ will always denote a one-dimensional **arithmetic scheme**; that is, separated scheme of finite type $X \to \operatorname{Spec} \mathbb{Z}$ of Krull dimension 1. The number $n$ will be a fixed *strictly negative* integer.

**Motivic cohomology.** We will work with étale motivic cohomology defined in terms of Bloch's cycle complexes. These were introduced by Bloch in [Blo1986] for varieties over fields, and for the version over $\operatorname{Spec} \mathbb{Z}$ that we will use, see [Gei2004, Gei2005].

Namely, we let $\Delta^i = \operatorname{Spec} \mathbb{Z}[t_0, \ldots, t_i]/(1 - \sum_i t_i)$ be the algebraic simplex. Denote by $z_n(X, i)$ the group freely generated by algebraic cycles $Z \subset X \times \Delta^i$ of dimension $n + i$. Then for $n < 0$ we let $\mathbb{Z}^c(n)$ be the complex of étale sheaves $U \rightsquigarrow z_n(U, -\bullet)[2n]$. The corresponding (hyper)cohomology $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ is what we will call motivic cohomology throughout this text. For a proper regular arithmetic scheme $X$ of dimension $d$ we have $\mathbb{Z}^c(n) = \mathbb{Z}(d - n)[2d]$, where $\mathbb{Z}(m)$ is the other motivic complex that usually appears in the literature.

We recall from [Gei2010, Corollary 7.2] that our motivic cohomology satisfies the localization property: if $Z \subset X$ is a closed subscheme and $U = X \setminus Z$ is its closed complement, then there is a distinguished triangle

$$R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n)) \to R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)) \to R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)) \to R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n))[1],$$

which gives a long exact sequence

$$\cdots \to H^i(Z_{\acute{e}t}, \mathbb{Z}^c(n)) \to H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) \to H^i(U_{\acute{e}t}, \mathbb{Z}^c(n)) \to H^{i+1}(Z_{\acute{e}t}, \mathbb{Z}^c(n)) \to \cdots \tag{1.6}$$

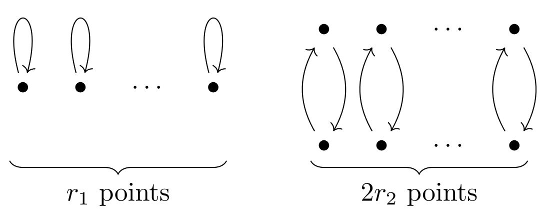
This means that $H^i(-, \mathbb{Z}^c(n))$ behaves like (motivic) Borel–Moore homology. For the corresponding zeta functions, we have

$$\zeta(X, s) = \zeta(Z, s)\, \zeta(U, s)$$

—this is clear from the definition.

In general, the groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are very difficult to compute. However, these are rather well-understood for one-dimensional arithmetic schemes $X$; see §3 below.

**Real and complex places.** Consider the finite discrete space of complex points $X(\mathbb{C}) = \operatorname{Hom}(\operatorname{Spec} \mathbb{C}, X)$. There is a canonical action of complex conjugation $G_\mathbb{R} := \operatorname{Gal}(\mathbb{C}/\mathbb{R})$ on $X(\mathbb{C})$. The fixed points of this action correspond to the real points $X(\mathbb{R})$. We will denote $r_1 = |X(\mathbb{R})|$. The non-real points come in conjugate pairs, and we will denote their number by $2r_2$.




Equivalently, for a number field $F/\mathbb{Q}$, denote by $r_1(F)$ the number of real embeddings $F \hookrightarrow \mathbb{R}$ and and by $r_2(F)$ the number of pairs of complex embeddings $F \hookrightarrow \mathbb{C}$. Then $r_1(F) = r_1$ and $r_2(F) = r_2$ for $X = \operatorname{Spec} \mathcal{O}_F$. In general, for a one-dimensional arithmetic scheme $X$ we have

$$r_1 = \sum_{\operatorname{char} \kappa(\eta) = 0} r_1(\kappa(\eta)),$$
$$r_2 = \sum_{\operatorname{char} \kappa(\eta) = 0} r_2(\kappa(\eta)),$$

where the sums are over generic points $\eta \in X$ with residue field $\kappa(\eta)$ of characteristic 0.

It will be convenient to introduce the following notation.

**1.3. Definition.** If $X$ is a one-dimensional arithmetic scheme with $r_1$ real and $2r_2$ complex places, we define for $n < 0$

$$d_n = \begin{cases} r_1 + r_2, & n \text{ even}, \\ r_2, & n \text{ odd}. \end{cases}$$

There is no coincidence that this notation coincides with (1.1); we will prove this in proposition 3.1 below.

**Abelian groups.** In all our calculations, an arithmetic duality [Bes2020, Theorem I] will play a major role. It affirms that certain groups are $\mathbb{Q}/\mathbb{Z}$-dual to finitely generated groups $H^i(X_{ét}, \mathbb{Z}^c(n))$. It will be convenient to fix some notation.

**1.4. Definition.** Given a finitely generated abelian group $A$, we denote

$$A^D := \operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z}), \quad A^* := \operatorname{Hom}(A, \mathbb{Z}).$$

**1.5. Remark.** We have the following immediate observations.

- There is an exact sequence $0 \to A^* \to \operatorname{Hom}(A, \mathbb{Q}) \to A^D \to (A_{tors})^D \to 0$
- $A^*$ is a free group, and $\operatorname{rk}_{\mathbb{Z}} A^* = \operatorname{rk}_{\mathbb{Z}} A$.
- If $A$ is finite, then there is a *non-canonical* isomorphism $A^D \cong A$, and in particular $|A^D| = |A|$.

## Outline of the paper

In §2 we prove a dévissage lemma, which shows how a property that holds for curves over finite fields and for number rings may be generalized to any one-dimensional arithmetic scheme. This is an elementary argument, but it is isolated in order not to repeat the same reasoning in several proofs.

Then in §3 we put together various well-known results in order to to describe motivic cohomology groups $H^i(X_{ét}, \mathbb{Z}^c(n))$ for one-dimensional arithmetic schemes.

The following §4 is dedicated to an explicit description of Weil-étale cohomology groups $H^i_{W,c}(X, \mathbb{Z}(n))$ that were defined in [Bes2020], again for the particular case of one-dimensional $X$. In §5 we define the regulator, and in §6 the calculations of Weil-étale cohomology are used to write down an explicit formula for the special value $\zeta^*(X, n)$, which corresponds to the conjecture $\mathbf{C}(X, n)$ from [Bes2021]. The main theorem is then deduced from the results of [Bes2021].

In §7 it is explained how to prove the formula directly, using localization. This is essentially the argument from [Bes2021], only written out explicitly for one-dimensional $X$.

We conclude in §8 with a couple of examples that illustrate how the special value formula works.

## Acknowledgments

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# 2 Dévissage lemma

The main idea of this text is to consider a property that holds for spectra of number rings $X = \operatorname{Spec} \mathcal{O}_F$ and curves over finite fields $X/\mathbb{F}_q$, and then formally generalize it to any one-dimensional arithmetic scheme. For this we isolate in this section a dévissage argument, which will be used repeatedly in everything that follows.

**2.1. Lemma.** *Let $\mathcal{P}$ be a property of arithmetic schemes of Krull dimension $\le 1$. Assume that it satisfies the following compatibilities.*

*a) $\mathcal{P}(X)$ holds if and only if $\mathcal{P}(X_{red})$ does.*

*b) If $X = \coprod_i X_i$ is a finite disjoint union, then $\mathcal{P}(X)$ is equivalent to the conjunction of $\mathcal{P}(X_i)$ for all $i$.*

*c) If $U \subset X$ is a dense open subset, then $\mathcal{P}(X)$ is equivalent to $\mathcal{P}(U)$.*

*Suppose that*

*0)* $\mathcal{P}(\operatorname{Spec}\mathbb{F}_q)$ *holds for any finite field* $\mathbb{F}_q$*,*

*1)* $\mathcal{P}(X)$ *holds for any smooth curve* $X/\mathbb{F}_q$*,*

*2)* $\mathcal{P}(\operatorname{Spec}\mathcal{O}_F)$ *holds for any number field* $F/\mathbb{Q}$*.*

*Then* $\mathcal{P}(X)$ *holds for any one-dimensional arithmetic scheme* $X$*.*

*Proof.* First suppose that $\dim X = 0$. Then thanks to a), we may suppose that $X$ is reduced, and then $X = \coprod_i \operatorname{Spec}\mathbb{F}_{q,i}$ is a finite disjoint union of spectra of finite fields, so that $\mathcal{P}(X)$ holds thanks to 0) and b).

Now consider the case of $\dim X = 1$. Again, we may assume that $X$ is reduced. Consider the normalization $\nu\colon X' \to X$. This is a birational morphism: there exist dense open subsets $U \subset X$ and $U' \subset X'$ such that $\nu|_{U'} : U' \xrightarrow{\cong} U$ is an isomorphism. Thanks to c), we have

$$\mathcal{P}(X) \iff \mathcal{P}(U) \iff \mathcal{P}(U') \iff \mathcal{P}(X').$$

Therefore, we may assume that $X$ is regular. Now $X = \coprod_i X_i$ is a finite disjoint union of normal integral schemes, so thanks to b), we may assume that $X$ is integral. There are two cases.

- If $X \to \operatorname{Spec}\mathbb{Z}$ lives over a closed point, then it is a smooth curve over $\mathbb{F}_q$, and the claim holds thanks to 1).
- If $X \to \operatorname{Spec}\mathbb{Z}$ is a dominant morphism, consider an open affine neighborhood of the generic point $U \subset X$. Again, $\mathcal{P}(X)$ is equivalent to $\mathcal{P}(U)$, so it will be enough to prove the claim for $U$. We have $U = \operatorname{Spec}\mathcal{O}_{F,S}$ for some number field $F/\mathbb{Q}$ and a finite set of places $S$, hence everything again reduces to $\mathcal{P}(\operatorname{Spec}\mathcal{O}_F)$. □

# 3 Calculations of motivic cohomology

In this section we review some results regarding the structure of étale motivic cohomology $H^i(X_{ét}, \mathbb{Z}^c(n))$. What follows is rather well-known, but we put together the references and state the corresponding results for a general one-dimensional arithmetic scheme.

## Vanishing order of $\zeta(X,s)$ at $s = n < 0$

In the subsequent calculations the number $d_n$ from definition 1.3 will appear frequently, so we make a digression to explain its arithmetic meaning.

**3.1. Proposition.** *We have* $d_n = \operatorname{ord}_{s=n}\zeta(X,s)$*.*

*Proof.* For $X = \operatorname{Spec}\mathcal{O}_F$ the claim is a consequence of the functional equation for the Dedekind zeta function [Neu1999, §VII.5]. It is also true for $X/\mathbb{F}_q$ since in this case $\zeta(X,s)$ has no zeros or poles at $s = n < 0$ [Kat1994, pp. 26–27]. We proceed with dévissage lemma 2.1.

We have $\zeta(X,s) = \zeta(X_{red},s)$ and $r_{1,2}(X) = r_{1,2}(X_{red})$. If $X = \coprod_i X_i$ is a finite disjoint union, then

$$\operatorname{ord}_{s=n}\zeta(X,s) = \sum_i \operatorname{ord}_{s=n}\zeta(X_i,s), \quad r_{1,2}(X) = \sum_i r_{1,2}(X_i),$$

so that the property is compatible with disjoint unions. Finally, if $U \subset X$ is a dense open subset, then $Z = X \setminus U$ is a zero-dimensional scheme, and

$$\operatorname{ord}_{s=n}\zeta(X,s) = \operatorname{ord}_{s=n}\zeta(U,s), \quad r_{1,2}(X) = r_{1,2}(U),$$

so that the property is compatible with taking dense open subsets. We conclude that lemma 2.1 applies. □

### $G_{\mathbb{R}}$-equivariant cohomology of $X(\mathbb{C})$

Viewing $\mathbb{Z}(n) = (2\pi i)^n\,\mathbb{Z}$ as a constant $G_{\mathbb{R}}$-equivariant sheaf on $X(\mathbb{C})$, we consider the $G_{\mathbb{R}}$-equivariant cohomology groups (resp. Tate cohomology)

$$H^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) := H^i\Big(R\Gamma(G_{\mathbb{R}}, R\Gamma_c(X(\mathbb{C}), \mathbb{Z}(n)))\Big),$$
$$\widehat{H}^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) := H^i\Big(R\widehat{\Gamma}(G_{\mathbb{R}}, R\Gamma_c(X(\mathbb{C}), \mathbb{Z}(n)))\Big).$$

Of course, $X(\mathbb{C})$ is just a finite discrete space, so that there is no need to use cohomology with compact support $H^i_c$ and $\widehat{H}^i_c$, but we will use this notation for consistency with the general case, treated in [Bes2020]. Since $\dim X(\mathbb{C}) = 0$, we have

$$H^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \cong H^i(G_{\mathbb{R}}, H^0_c(X(\mathbb{C}), \mathbb{Z}(n))),$$
$$\widehat{H}^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \cong \widehat{H}^i(G_{\mathbb{R}}, H^0_c(X(\mathbb{C}), \mathbb{Z}(n))).$$

**3.2. Proposition.** *Let $X$ be a one-dimensional arithmetic scheme with $r_1$ real and $2r_2$ complex places. Then the $G_{\mathbb{R}}$-equivariant cohomology of $X(\mathbb{C})$ is given by*

$$\widehat{H}^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\oplus r_1}, & i \equiv n\ (2), \\ 0, & i \not\equiv n\ (2); \end{cases}$$
$$H^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \cong \begin{cases} 0, & i < 0, \\ \mathbb{Z}^{\oplus d_n}, & i = 0, \\ \widehat{H}^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)), & i \ge 1. \end{cases}$$

*Proof.* We have
$$H^0_c(X(\mathbb{C}), \mathbb{Z}(n)) \cong \mathbb{Z}(n)^{\oplus r_1} \oplus (\mathbb{Z}(n) \oplus \mathbb{Z}(n))^{\oplus r_2},$$
and the $G_{\mathbb{R}}$-action on the two summands is given by $x \mapsto \overline{x}$ and $(x, y) \mapsto (\overline{y}, \overline{x})$ respectively. We recall that Tate cohomology of a finite cyclic group is 2-periodic:

$$\widehat{H}^i(G, A) \cong \begin{cases} \widehat{H}^0(G, A), & i \text{ even}, \\ \widehat{H}_0(G, A), & i \text{ odd}, \end{cases}$$

and the groups $\widehat{H}^0(G, A)$ and $\widehat{H}_0(G, A)$ are given by the exact sequence

$$0 \to \widehat{H}_0(G, A) \to A_G \xrightarrow{N} A^G \to \widehat{H}^0(G, A) \to 0$$

where $N\colon A_G \to A^G$ is the norm map, induced by the action of $\sum_{g \in G} g$.

Therefore, we may consider two cases.

1) $G_{\mathbb{R}}$-module $A = \mathbb{Z}(n)$ with action via $x \mapsto \overline{x}$. In this case we see that

$$A^{G_{\mathbb{R}}} \cong \begin{cases} \mathbb{Z}, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

Similarly it is easy to calculate the coinvariants $A_{G_{\mathbb{R}}}$, and

$$\widehat{H}^0(G_{\mathbb{R}}, A) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z}, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \quad \widehat{H}_0(G_{\mathbb{R}}, A) \cong \begin{cases} 0, & n \text{ even}, \\ \mathbb{Z}/2\mathbb{Z}, & n \text{ odd}. \end{cases}$$

2) $G_\mathbb{R}$-module $A = \mathbb{Z}(n) \oplus \mathbb{Z}(n)$ with action via $(x, y) \mapsto (\overline{y}, \overline{x})$. In this case $A^{G_\mathbb{R}} \cong \mathbb{Z}$ and $\widehat{H}^0(G_\mathbb{R}, A) = \widehat{H}_0(G_\mathbb{R}, A) = 0$.

Putting together these two calculations, we obtain Tate cohomology groups $\widehat{H}^i_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))$. For the usual cohomology, we have

$$\begin{aligned} H^0_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) &\cong H^0_c(X(\mathbb{C}), \mathbb{Z}(n))^{G_\mathbb{R}}, \\ H^i_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) &\cong \widehat{H}^i_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \quad \text{for } i \geq 1. \end{aligned}$$

□

## Étale motivic cohomology

We now describe motivic cohomology groups $H^i(X_{ét}, \mathbb{Z}^c(n))$ for one-dimensional arithmetic schemes.

**3.3. Proposition.** *If $X$ is a one-dimensional arithmetic scheme and $n < 0$, then*

$$H^i(X_{ét}, \mathbb{Z}^c(n)) \cong \begin{cases} 0, & i < -1, \\ \text{finitely generated of rk } d_n, & i = -1, \\ \text{finite}, & i = 0, 1, \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus r_1}, & i \geq 2, \ i \not\equiv n \ (2), \\ 0, & i \geq 2, \ i \equiv n \ (2). \end{cases} \tag{3.1}$$

*Further, if $X = \operatorname{Spec} \mathcal{O}_F$ for a number field $F/\mathbb{Q}$, then*

$$H^1(X_{ét}, \mathbb{Z}^c(n)) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\oplus r_1}, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases} \tag{3.2}$$

An important ingredient of the proof, and other arguments below, will be an arithmetic duality [Bes2020, Theorem I], which states that if $H^i(X_{ét}, \mathbb{Z}^c(n))$ are finitely generated groups for all $i \in \mathbb{Z}$, then

$$\widehat{H}^i_c(X_{ét}, \mathbb{Z}(n)) \cong H^{2-i}(X_{ét}, \mathbb{Z}^c(n))^D, \tag{3.3}$$

where

$$\mathbb{Z}(n) := \mathbb{Q}/\mathbb{Z}(n)[-1] := \bigoplus_p \varinjlim_r j_{p!} \mu_{p^r}^{\otimes n}[-1].$$

Here $\widehat{H}^i_c(X_{ét}, \mathbb{Z}(n))$ is the modified cohomology with compact support, for which we refer to [Bes2020, Appendix B] or [GS2018, §2]. In particular, $\widehat{H}^i_c(X_{ét}, \mathbb{Z}(n)) = H^i_c(X_{ét}, \mathbb{Z}(n))$ whenever $X(\mathbb{R}) = \varnothing$. We recall from definition 1.4 that $A^D$ denotes $\operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z})$. We note that (3.3) is a powerful result, and it is deduced in [Bes2020] from Geisser's work [Gei2010].

*Proof of proposition 3.3.* We will use the dévissage lemma 2.1. We will say that $\mathcal{P}(X)$ holds if the motivic cohomology of $X$ has structure (3.1).

**First consider the case of a finite field** $X = \operatorname{Spec} \mathbb{F}_q$. We have

$$H^i(\operatorname{Spec} \mathbb{F}_{q,ét}, \mathbb{Z}^c(n)) \cong \begin{cases} \mathbb{Z}/(q^{-n} - 1), & i = 1, \\ 0, & i \neq 1. \end{cases} \tag{3.4}$$

—see for instance [Gei2017, Example 4.2]. This is related to Quillen's calculation of $K$-theory of finite fields [Qui1972].

In general, if $X$ is a zero-dimensional arithmetic scheme, then the motivic cohomology coincides for $X$ and $X_{red}$, hence we may assume that $X$ is reduced. Then $X$ is a finite disjoint union of $X_i = \operatorname{Spec} \mathbb{F}_{q_i}$, and

$$H^i(X, \mathbb{Z}^c(n)) = \begin{cases} \text{finite}, & i = 1, \\ 0, & i \neq 1. \end{cases} \tag{3.5}$$

In particular, $\mathcal{P}(X)$ holds if $\dim X = 0$.

**Now we check the compatibility properties for** $\mathcal{P}$. If $X = \coprod_i X_i$ is a finite disjoint union, then $H^i(X_{ét}, \mathbb{Z}^c(n)) \cong \bigoplus_i H^i(X_{i,ét}, \mathbb{Z}^c(n))$, so that the property $\mathcal{P}$ is compatible with disjoint unions.

Similarly, if $U \subset X$ is a dense open subset, and $Z = X \setminus U$ its closed complement, then $\dim Z = 0$. We consider the long exact sequence (1.6). Since cohomology of $Z$ is concentrated in $i = 1$, we have $H^i(X_{ét}, \mathbb{Z}^c(n)) \cong H^i(U_{ét}, \mathbb{Z}^c(n))$ for $i \neq 0, 1$, and what remains is an exact sequence

$$0 \to H^0(X_{ét}, \mathbb{Z}^c(n)) \to H^0(U_{ét}, \mathbb{Z}^c(n)) \to H^1(Z_{ét}, \mathbb{Z}^c(n)) \to H^1(X_{ét}, \mathbb{Z}^c(n)) \to H^1(U_{ét}, \mathbb{Z}^c(n)) \to 0$$

Moreover, $d_n(X) = d_n(U)$. These considerations show that $\mathcal{P}(X)$ and $\mathcal{P}(U)$ are equivalent, and therefore lemma 2.1 works, and it remains to establish $\mathcal{P}(X)$ for a curve $X/\mathbb{F}_q$ or $X = \operatorname{Spec} \mathcal{O}_F$.

**Suppose that** $X/\mathbb{F}_q$ **is a smooth curve**. The groups $H^i(X_{ét}, \mathbb{Z}^c(n))$ are finitely generated by [Gei2017, Proposition 4.3], so that the duality (3.3) holds. The $\mathbb{Q}/\mathbb{Z}$-dual groups

$$H^i_c(X_{ét}, \mathbb{Z}(n)) = \bigoplus_\ell H^{i-1}_c(X_{ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$$

are finite by [Kah2003, Theorem 3], and concentrated in $i = 1, 2, 3$ for dimension reasons. It follows that in this case $H^i(X_{ét}, \mathbb{Z}^c(n))$ are finite groups concentrated in $i = -1, 0, 1$, and the property $\mathcal{P}(X)$ holds.

**It remains to consider the case of** $X = \operatorname{Spec} \mathcal{O}_F$. In this case finite generation of $H^i(X_{ét}, \mathbb{Z}^c(n))$ is also known; see for instance [Gei2017, Proposition 4.14]. Therefore, the duality (3.3) holds. We have $\widehat{H}^i_c(\operatorname{Spec} \mathcal{O}_F[1/p], \mu_{p^r}^{\otimes n}) = 0$ for $i \geq 3$ by Artin–Verdier duality [Mil2006, Chapter II, Corollary 3.3], or by [Sou1979, p. 268]. Therefore, it follows that $\widehat{H}^i_c(X_{ét}, \mathbb{Z}(n)) = 0$ for $i \geq 4$, and hence by duality (3.3), $H^i(X_{ét}, \mathbb{Z}^c(n)) = 0$ for $i \leq -2$.

Now we identify the finite 2-torsion in $H^i(X_{ét}, \mathbb{Z}^c(n))$ for $i \geq 2$. By [FM2018, Lemma 6.14], there is an exact sequence

$$\cdots \to H^{i-1}_c(X_{ét}, \mathbb{Z}(n)) \to \widehat{H}^{i-1}(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \to \widehat{H}^i_c(X_{ét}, \mathbb{Z}(n)) \to H^i_c(X_{ét}, \mathbb{Z}(n)) \to \cdots \tag{3.6}$$

For $i \leq 0$ we have $H^i_c(X_{ét}, \mathbb{Z}(n)) = 0$, and therefore

$$\widehat{H}^i_c(X_{ét}, \mathbb{Z}(n)) \cong \widehat{H}^{i-1}(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\oplus r_1}, & i \not\equiv n \ (2), \\ 0, & i \equiv n \ (2). \end{cases}$$

By duality, we have for $i \geq 2$

$$H^i(X_{ét}, \mathbb{Z}^c(n)) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\oplus r_1}, & i \not\equiv n \ (2), \\ 0, & i \equiv n \ (2). \end{cases}$$

Now we determine the ranks of $H^i(X_{ét}, \mathbb{Z}^c(n))$ for $i = -1, 0, 1$. Put $m = 1 - n$, so that $m \geq 2$. We have for $i = -1, 0, 1$

$$H^i(X_{ét}, \mathbb{Z}^c(n)) \cong H^{2+i}(X_{ét}, \mathbb{Z}(m)) \cong H^{2+i}(X_{Zar}, \mathbb{Z}(m)).$$

Here the last isomorphism is Beilinson–Lichtenbaum conjecture, which is now a theorem [Gei2004, Theorem 1.2]. We will use this identification to analyze the groups $H^i(X_{ét}, \mathbb{Z}(m))$ for $i = 1, 2, 3$. According to

[KS2008, Proposition 2.1], for $i = 1, 2$ the Chern character $K_{2m-i}(X) \to H^i(X_{Zar}, \mathbb{Z}(m))$ has finite 2-torsion kernel and cokernel. In particular, $\operatorname{rk}_{\mathbb{Z}} H^i(X_{Zar}, \mathbb{Z}(m)) = \operatorname{rk}_{\mathbb{Z}} K_{2m-i}(X)$. The ranks of $K$-theory of rings of integers were calculated by Borel [Bor1974]:

$$\operatorname{rk} K_{2m-i}(X) = \begin{cases} 0, & 2m - i \text{ even}, \\ r_1 + r_2, & 2m - i \equiv 1 \ (4), \\ r_2, & 2m - i \equiv 3 \ (4). \end{cases}$$

This implies that $H^2(X_{Zar}, \mathbb{Z}(m))$ is a finite group, while

$$\operatorname{rk}_{\mathbb{Z}} H^1(X_{Zar}, \mathbb{Z}(m)) = \begin{cases} r_2, & m \text{ even}, \\ r_1 + r_2, & m \text{ odd}. \end{cases}$$

Finally, for $i = 3$ we have by [KS2008, p. 179]

$$H^i(X_{Zar}, \mathbb{Z}(m)) \cong \begin{cases} 0, & m \text{ even}, \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus r_1}, & m \text{ odd}. \end{cases}$$

This concludes the proof. □

# 4 Calculations of Weil-étale cohomology

In this section we calculate Weil-étale cohomology groups $H^i_{W,c}(X, \mathbb{Z}(n))$ for $n < 0$, as defined in [Bes2020]. We briefly recall the construction. Let $X$ be an arithmetic scheme with finitely generated motivic cohomology $H^i(X_{ét}, \mathbb{Z}^c(n))$. The construction is performed in two steps.

- **Step 1**. Consider the morphism in the derived category $\mathbf{D}(\mathbb{Z})$

$$\alpha_{X,n} \colon R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \to R\Gamma_c(X_{ét}, \mathbb{Z}(n))$$

determined on the level of cohomology, using the arithmetic duality (3.3), via

$$H^i(\alpha_{X,n}) \colon \mathrm{Hom}(H^{2-i}(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}) \xrightarrow{\mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}} H^{2-i}(X_{ét}, \mathbb{Z}^c(n))^D \xleftarrow{\cong} \widehat{H}^i_c(X_{ét}, \mathbb{Z}(n)) \to H^i_c(X_{ét}, \mathbb{Z}(n)).$$

The complex $R\Gamma_{fg}(X, \mathbb{Z}(n))$ is defined as a cone of $\alpha_{X,n}$:

$$R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \xrightarrow{\alpha_{X,n}} R\Gamma_c(X_{ét}, \mathbb{Z}(n)) \to R\Gamma_{fg}(X_{ét}, \mathbb{Z}(n)) \to R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-1])$$

The groups $H^i_{fg}(X, \mathbb{Z}(n)) := H^i(R\Gamma_{fg}(X, \mathbb{Z}(n)))$ are finitely generated for all $i \in \mathbb{Z}$, vanish for $i \ll 0$, and finite 2-torsion for $i \gg 0$. We refer to [Bes2020, §5] for the details.

- **Step 2**. One considers a canonical morphism $i_\infty^*$ in the derived category $\mathbf{D}(\mathbb{Z})$ that is torsion and gives a commutative diagram

$$\begin{array}{ccc} R\Gamma_c(X_{ét}, \mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X, \mathbb{Z}(n)) \\ {\scriptstyle u_\infty^*}\downarrow & \swarrow {\scriptstyle i_\infty^*} & \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) & & \end{array}$$

—see [Bes2020, §§6,7] for further details. Weil-étale cohomology with compact support is defined as a mapping fiber of $i_\infty^*$:

$$R\Gamma_{W,c}(X,\mathbb{Z}(n)) \to R\Gamma_{fg}(X,\mathbb{Z}(n)) \xrightarrow{i_\infty^*} R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{Z}(n)) \to R\Gamma_{W,c}(X,\mathbb{Z}(n))[1]$$

The resulting groups $H^i_{W,c}(X,\mathbb{Z}(n)) := H^i(R\Gamma_{W,c}(X,\mathbb{Z}(n)))$ are finitely generated and vanish for $i \notin [0,2d+1]$. We refer to [Bes2020, §7] for the general properies.

Here we calculate $H^i_{W,c}(X,\mathbb{Z}(n))$ for one-dimensional $X$.

**4.1. Proposition.** *Let $X$ be a one-dimensional arithmetic scheme and $n<0$.*

*0) $H^i_{W,c}(X,\mathbb{Z}(n)) = 0$, unless $i = 1,2,3$.*

*1) There is a short exact sequence*

$$0 \to \underbrace{H^0_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{Z}(n))}_{\cong\mathbb{Z}^{\oplus d_n}} \to H^1_{W,c}(X,\mathbb{Z}(n)) \to T_1 \to 0 \tag{4.1}$$

*where $T_1$ sits in a short exact sequence of finite groups*

$$0 \to \widehat{H}^0_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{Z}(n)) \to H^1(X_{ét},\mathbb{Z}^c(n))^D \to T_1 \to 0$$

*In particular, $H^1_{W,c}(X,\mathbb{Z}(n))$ is finitely generated of rank $d_n$, and*

$$|T_1| = \frac{1}{2^\delta}\cdot |H^1(X_{ét},\mathbb{Z}^c(n))|,$$

*where $\delta$ is defined by (1.5).*

*2) There is an isomorphism of finitely generated groups*

$$H^2_{W,c}(X,\mathbb{Z}(n)) \cong \underbrace{H^{-1}(X_{ét},\mathbb{Z}^c(n))^*}_{\cong\mathbb{Z}^{\oplus d_n}} \oplus \underbrace{H^0(X_{ét},\mathbb{Z}^c(n))^D}_{finite}.$$

*3) There is an isomorphism of finite groups*

$$H^3_{W,c}(X,\mathbb{Z}(n)) \cong (H^{-1}(X_{ét},\mathbb{Z}^c(n))_{tors})^D.$$

We recall from definition 1.4 that $A^D := \operatorname{Hom}(A,\mathbb{Q}/\mathbb{Z})$ and $A^* := \operatorname{Hom}(A,\mathbb{Z})$.

*Proof.* From the definition of $R\Gamma_{fg}(X,\mathbb{Z}(n))$, we have a long exact sequence

$$\cdots \to \operatorname{Hom}(H^{2-i}(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}) \xrightarrow{H^i(\alpha_{X,n})} H^i_c(X_{ét},\mathbb{Z}(n)) \to H^i_{fg}(X,\mathbb{Z}(n)) \to \operatorname{Hom}(H^{1-i}(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}) \to \cdots \tag{4.2}$$

From our calculations of motivic cohomology (proposition 3.3), we have $\operatorname{Hom}(H^i(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}) = 0$, unless $i=-1$, and $H^i_c(X_{ét},\mathbb{Z}(n)) = 0$ for $i\le 0$. This implies that $H^i_{fg}(X,\mathbb{Z}(n)) = 0$ for $i \le 0$. As $H^i_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{Z}(n)) = 0$ for $i<0$, we see from the exact sequence

$$\cdots \to H^i_{W,c}(X,\mathbb{Z}(n)) \to H^i_{fg}(X,\mathbb{Z}(n)) \xrightarrow{H^i(i_\infty^*)} H^i_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{Z}(n)) \to H^{i+1}_{W,c}(X,\mathbb{Z}(n)) \to \cdots \tag{4.3}$$

that $H^i_{W,c}(X,\mathbb{Z}(n)) = 0$ for $i\le 0$.

For $i = 1$, the exact sequence (4.2) shows that $H^1_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to H^1_{fg}(X_{\acute{e}t}, \mathbb{Z}(n))$ is an isomorphism. Consequently, we see that $\ker H^1(i^*_\infty) \cong \ker H^1(u^*_\infty)$:

$$\begin{array}{ccc} H^1_c(X_{\acute{e}t}, \mathbb{Z}(n)) & \xrightarrow{\cong} & H^1_{fg}(X, \mathbb{Z}(n)) \\ {\scriptstyle H^1(u^*_\infty)}\downarrow & \swarrow {\scriptstyle H^1(i^*_\infty)} & \\ H^1_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) & & \end{array}$$

From long exact sequences (4.3) and (3.6), we obtain short exact sequences

$$0 \to H^0_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \to H^1_{W,c}(X, \mathbb{Z}(n)) \to \ker H^1(i^*_\infty) \to 0$$
$$0 \to \widehat{H}^0_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \to \widehat{H}^1_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to \ker H^1(u^*_\infty) \to 0$$

As $\ker H^1(i^*_\infty) \cong \ker H^1(u^*_\infty)$, this is part 1) of the proposition.

We proceed with calculation of $H^i_{W,c}(X, \mathbb{Z}(n))$ for $i \geq 2$. It is more convenient to do that without passing through $H^i_{fg}(X, \mathbb{Z}(n))$ explicitly. Consider the morphism of complexes

$$\widehat{\alpha}_{X,n} \colon R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \to R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n)),$$

defined in the same way as $\alpha_{X,n}$, only without taking the projection $R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n))$:

$$H^i(\widehat{\alpha}_{X,n}) \colon \mathrm{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) \xrightarrow{\mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}} H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))^D \xleftarrow{\cong} \widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)).$$

The relation between $\widehat{\alpha}_{X,n}$ and $\alpha_{X,n}$ is given by

$$\begin{array}{ccc} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & \xrightarrow{\widehat{\alpha}_{X,n}} & R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n)) \\ & \searrow {\scriptstyle \alpha_{X,n}} & \downarrow \\ & & R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \end{array}$$

Here the vertical arrow comes from the definition of modified étale cohomology with compact support, and it sits in an exact triangle

$$R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \xrightarrow{\widehat{u}^*_\infty} R\widehat{\Gamma}_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \to R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n))[1]$$

—see [FM2018, Lemma 6.14]. From the definition of $\widehat{\alpha}_{X,n}$, we calculate (see remark 1.5)

$$\ker H^i(\widehat{\alpha}_{X,n}) = H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))^*,$$
$$\mathrm{coker}\, H^i(\widehat{\alpha}_{X,n}) \cong (H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{tors})^D.$$

We denote a cone of $\widehat{\alpha}_{X,n}$ by $R\widehat{\Gamma}_{fg}(X, \mathbb{Z}(n))$, and put $\widehat{H}^i_{fg}(X, \mathbb{Z}(n)) = H^i(R\widehat{\Gamma}_{fg}(X, \mathbb{Z}(n)))$, so that there is a long exact sequence

$$\cdots \to \mathrm{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) \xrightarrow{H^i(\widehat{\alpha}_{X,n})}$$
$$\widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to \widehat{H}^i_{fg}(X, \mathbb{Z}(n)) \to \mathrm{Hom}(H^{1-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) \to \cdots$$

The corresponding short exact sequences

$$0 \to \mathrm{coker}\, H^i(\widehat{\alpha}_{X,n}) \to \widehat{H}^i_{fg}(X, \mathbb{Z}(n)) \to \ker H^{i+1}(\widehat{\alpha}_{X,n}) \to 0$$

are split, since $\ker H^{i+1}(\widehat{\alpha}_{X,n})$ is a free group. Therefore, we have

$$\widehat{H}^i_{fg}(X,\mathbb{Z}(n)) \cong H^{1-i}(X_{\acute{e}t},\mathbb{Z}^c(n))^* \oplus (H^{2-i}(X_{\acute{e}t},\mathbb{Z}^c(n))_{tors})^D.$$

There is a commutative diagram with distinguished rows and columns

$$\begin{array}{ccccccc}
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]) & \xrightarrow{\widehat{\alpha}_{X,n}} & R\widehat{\Gamma}_c(X_{\acute{e}t},\mathbb{Z}(n)) & \longrightarrow & R\widehat{\Gamma}_{fg}(X,\mathbb{Z}(n)) & \longrightarrow & [+1] \\
\downarrow{\scriptstyle id} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id} \\
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]) & \xrightarrow{\alpha_{X,n}} & R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X,\mathbb{Z}(n)) & \longrightarrow & [+1] \\
\downarrow & & \downarrow{\scriptstyle \widehat{u}^*_\infty} & & \downarrow{\scriptstyle \widehat{i}^*_\infty} & & \downarrow \\
0 & \longrightarrow & R\widehat{\Gamma}_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{Z}(n)) & \xrightarrow{id} & R\widehat{\Gamma}_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{Z}(n)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]) & \xrightarrow{\widehat{\alpha}_{X,n}[1]} & R\widehat{\Gamma}_c(X_{\acute{e}t},\mathbb{Z}(n))[1] & \longrightarrow & R\widehat{\Gamma}_{fg}(X,\mathbb{Z}(n))[1] & \longrightarrow & [+2]
\end{array}$$

Here $\widehat{u}^*_\infty$ (resp. $\widehat{i}^*_\infty$) is defined as the composition of the canonical morphism $u^*_\infty$ (resp. $i^*_\infty$) with the projection to Tate cohomology $\pi\colon R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{Z}(n)) \to R\widehat{\Gamma}_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{Z}(n))$.

In our particular case of $\dim X = 1$, we know that $H^i(\pi)$ is an isomorphism for $i \ge 1$. Therefore, the five-lemma applied to

$$\begin{array}{ccccccc}
R\Gamma_{W,c}(X,\mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X,\mathbb{Z}(n)) & \xrightarrow{i^*_\infty} & R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{Z}(n)) & \longrightarrow & R\Gamma_{W,c}(X,\mathbb{Z}(n))[1] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle f[1]} \\
R\widehat{\Gamma}_{fg}(X,\mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X,\mathbb{Z}(n)) & \xrightarrow{\widehat{i}^*_\infty} & R\widehat{\Gamma}_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{Z}(n)) & \longrightarrow & R\widehat{\Gamma}_{fg}(X,\mathbb{Z}(n))[1]
\end{array}$$

shows that for $i \ge 2$ holds

$$H^i_{W,c}(X,\mathbb{Z}(n)) \cong \widehat{H}^i_{fg}(X,\mathbb{Z}(n)) \cong H^{1-i}(X_{\acute{e}t},\mathbb{Z}^c(n))^* \oplus (H^{2-i}(X_{\acute{e}t},\mathbb{Z}^c(n))_{tors})^D.$$

Our calculations of motivic cohomology (proposition 3.3) show that $H^i_{W,c}(X,\mathbb{Z}(n)) = 0$ for $i \ge 4$, and

$$\begin{aligned}
H^2_{W,c}(X,\mathbb{Z}(n)) &\cong H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))^* \oplus H^0(X_{\acute{e}t},\mathbb{Z}^c(n))^D, \\
H^3_{W,c}(X,\mathbb{Z}(n)) &\cong (H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))_{tors})^D.
\end{aligned}$$

□

**4.2. Remark.** A priori, the short exact sequence (4.1) does not have to split. This will not bother us for the determinant calculations in §6 below.

**4.3. Remark.** Part 0) of the above proposition is a consequence of general results on boundedness of $H^i_{W,c}(X,\mathbb{Z}(n))$ that can be found in [Bes2020, §7].

**4.4. Remark.** The groups $H^i_{W,c}(X,\mathbb{Z}(n))$ for $X = \operatorname{Spec}\mathcal{O}_F$ are calculated in [FM2018, §5.8.3]. The result is (rewriting in terms of $H^i(X_{\acute{e}t},\mathbb{Z}^c(n)) \cong H^{2+i}(X_{\acute{e}t},\mathbb{Z}(1-n))$):

$$H^i_{W,c}(\operatorname{Spec}\mathcal{O}_F,\mathbb{Z}(n)) \cong \begin{cases} \mathbb{Z}^{\oplus d_n}, & i = 1, \\ H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))^* \oplus H^0(X_{\acute{e}t},\mathbb{Z}^c(n))^D, & i = 2, \\ (H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))_{tors})^D, & i = 3, \\ 0, & i \ne 1,2,3. \end{cases} \qquad (4.4)$$

Our calculation generalizes this. What may look puzzling is the general answer for $H^1_{W,c}(X,\mathbb{Z}(n))$ given by proposition 4.1. In case of $X = \operatorname{Spec}\mathcal{O}_F$, we have according to (3.2) that $H^1(X_{\acute{e}t},\mathbb{Z}^c(n)) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus r_1}$ for even $n$, and therefore $T_1 = 0$, which agrees with (4.4).

Intuitively, the arithmetically interesting cohomology $H^i(X_{ét}, \mathbb{Z}^c(n))$ for $X = \operatorname{Spec} \mathcal{O}_F$ is concentrated in degrees $i = -1, 0$. The groups $H^i(X_{ét}, \mathbb{Z}^c(n))$ for $i \ge 1$ bear no interesting information: these are finite 2-torsion, coming from the real places of $F$. Passing to Weil-étale cohomology removes this 2-torsion. On the other hand, for a curve over a finite field $X/\mathbb{F}_q$, the group $H^1(X_{ét}, \mathbb{Z}^c(n))$ is not trivial and bears arithmetic information. The finite group $T_1$ that appears in the statement corresponds to $H^1(X_{ét}, \mathbb{Z}^c(n))$, removing the 2-torsion coming from the real places of $X$.

**4.5. Remark.** For a curve over a finite field $X/\mathbb{F}_q$, all groups $H^i(X_{ét}, \mathbb{Z}^c(n))$ are finite, and our calculation gives $H^i_{W,c}(X, \mathbb{Z}(n)) \cong H^{2-i}(X_{ét}, \mathbb{Z}^c(n))^D$. This holds for any variety over a finite field $X/\mathbb{F}_q$, assuming finite generation of motivic cohomology $H^i(X_{ét}, \mathbb{Z}^c(n))$; see [Bes2020, Proposition 7.7].

**4.6. Remark.** It is conjectured in [Bes2021, §3] that

$$\operatorname{ord}_{s=n} \zeta(X, s) = \sum_{i \in \mathbb{Z}} (-1)^i \cdot i \cdot \operatorname{rk}_{\mathbb{Z}} H^i_{W,c}(X, \mathbb{Z}(n)).$$

In this case $\operatorname{rk}_{\mathbb{Z}} H^1_{W,c}(X, \mathbb{Z}(n)) = \operatorname{rk}_{\mathbb{Z}} H^2_{W,c}(X, \mathbb{Z}(n)) = d_n$ and $\operatorname{rk}_{\mathbb{Z}} H^3_{W,c}(X, \mathbb{Z}(n)) = 0$, so that the conjecture holds by proposition 3.1.

# 5 Regulators

Now we explain what will be intended by the regulator in our setting.

**5.1. Definition.** We let the **regulator morphism** be

$$\varrho_{X,n} \colon H^{-1}(X_{ét}, \mathbb{Z}^c(n)) \to H^{-1}(X_{ét}, \mathbb{Z}^c(n)) \otimes \mathbb{R} \xrightarrow{Reg_{X,n}} H^0_{BM}(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)),$$

where the map $Reg_{X,n}$ is defined in [Bes2021, §2].

The right hand side is Borel–Moore cohomology, defined via

$$H^0_{BM}(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)) := \operatorname{Hom}(H^0_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)), \mathbb{R}).$$

In general, the regulator takes values in Deligne–Beilinson cohomology, but the target simplifies in case of $n < 0$, as explained in [Bes2021, §2].

**5.2. Remark.** The only relevant group for the regulator is $H^{-1}(X_{ét}, \mathbb{Z}^c(n))$, since cohomology in other degrees is finite (see proposition 3.3).

**5.3. Remark.** If $X = \operatorname{Spec} \mathcal{O}_F$, then $\varrho_{X,n}$ is the usual Beilinson regulator map

$$H^1(X_{ét}, \mathbb{Z}(1-n)) \to H^1_{BM}(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(1-n))$$

that also appears in the special value conjecture in [FM2018, §5.8.3].

**5.4. Lemma.** *For any* 1*-dimensional scheme* $X$ *and* $n < 0$*, the* $\mathbb{R}$*-dual to the regulator*

$$Reg^\vee_{X,n} \colon H^0_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)) \to \operatorname{Hom}(H^{-1}(X_{ét}, \mathbb{Z}^c(n)), \mathbb{R})$$

*is an isomorphism.*

*Proof.* If $X/\mathbb{F}_q$, then the claim is trivial. For $X = \operatorname{Spec} \mathcal{O}_F$, this is a well-known property of Beilinson's regulator. To apply dévissage lemma 2.1, we need to check the compatibility with disjoint unions and taking a dense open subset $U \subset X$. For disjoint unions, this is clear. For a dense open subset $U \subset X$, the closed complement $Z = X \setminus U$ has dimension 0, and the localization exact sequence (1.6) with the long exact sequence for cohomology with compact support give integral isomorphisms

$$H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)) \xrightarrow{\cong} H^{-1}(U_{\acute{e}t}, \mathbb{Z}^c(n)),$$
$$H^0_c(G_\mathbb{R}, U(\mathbb{C}), \mathbb{Z}(n)) \xrightarrow{\cong} H^0_c(G_\mathbb{R}, Z(\mathbb{C}), \mathbb{Z}(n)).$$

We now have a commutative diagram

$$\begin{array}{ccc} H^0_c(G_\mathbb{R}, U(\mathbb{C}), \mathbb{R}(n)) & \xrightarrow{Reg^\vee_{U,n}} & \operatorname{Hom}(H^{-1}(U_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}) \\ \downarrow \cong & & \downarrow \cong \\ H^0_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n)) & \xrightarrow{Reg^\vee_{X,n}} & \operatorname{Hom}(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}) \end{array}$$

The top arrow is an isomorphism if and only if the bottom arrow is. □

**5.5. Definition.** For a one-dimensional arithmetic scheme $X$, we define the **regulator** as a positive real number

$$R_{X,n} := \operatorname{vol}\Bigl(\operatorname{coker}\bigl(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)) \xrightarrow{\varrho_{X,n}} H^0_{BM}(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n))\bigr)\Bigr)$$

where the volume is taken with respect to the canonical integral structure.

If $X(\mathbb{C}) = \varnothing$, or $n$ is odd and $r_2 = 0$, then $H^0_{BM}(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n)) = 0$, and we set $R_{X,n} = 1$.

**5.6. Lemma.** *Let $X$ be a one-dimensional scheme and $n < 0$. For any dense open subset $U \subset X$, one has $R_{X,n} = R_{U,n}$.*

*Proof.* Follows from the proof of lemma 5.4. □

**5.7. Proposition.** *Given an arithmetic scheme $X$ and $n < 0$, consider the two-term acyclic complex of real vector spaces*

$$C^\bullet\colon 0 \to \underbrace{H^0_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n))}_{\deg 0} \xrightarrow{Reg^\vee_{X,n}} \underbrace{\operatorname{Hom}(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})}_{\deg 1} \to 0$$

*Then taking the determinant $\det_\mathbb{R}(C^\bullet)$ in the sense of [KM1976], the image of the canonical map*

$$\det_\mathbb{Z} H^0_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \otimes_\mathbb{Z} \det_\mathbb{Z} \operatorname{Hom}(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z})^{-1} \to$$
$$\det_\mathbb{R} H^0_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n)) \otimes_\mathbb{R} \det_\mathbb{R} \operatorname{Hom}(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})^{-1} \xrightarrow{\cong} \mathbb{R}$$

*corresponds to $R_{X,n}\,\mathbb{Z} \subset \mathbb{R}$.*

*Proof.* In general, if $F$ and $F'$ are free groups of finite rank $d$, and

$$C^\bullet\colon 0 \to F \otimes_\mathbb{Z} \mathbb{R} \xrightarrow{\phi} F' \otimes_\mathbb{Z} \mathbb{R} \to 0$$

is a two-term acyclic complex of real vector spaces, then the image of

$$\mathbb{Z} \cong \det_\mathbb{Z} F \otimes_\mathbb{Z} (\det_\mathbb{Z} F')^{-1} \to \det_\mathbb{R}(F \otimes_\mathbb{Z} \mathbb{R}) \otimes_\mathbb{R} \det_\mathbb{R}(F' \otimes_\mathbb{Z} \mathbb{R})^{-1} = \det_\mathbb{R}(C^\bullet) \xrightarrow{\cong} \mathbb{R}$$

corresponds to $D\mathbb{Z} \subset \mathbb{R}$, where $D$ is the determinant of $\phi$ with respect to the bases induced by $\mathbb{Z}$-bases of $F$ and $F'$. This follows from the explicit description of the isomorphism $\det_\mathbb{R}(C^\bullet) \xrightarrow{\cong} \mathbb{R}$ from [KM1976, p. 33]: it is

$$\det_\mathbb{R}(F \otimes_\mathbb{Z} \mathbb{R}) \otimes_\mathbb{R} \det_\mathbb{R}(F' \otimes_\mathbb{Z} \mathbb{R})^{-1} \xrightarrow{\det_\mathbb{R}(\phi)} \det_\mathbb{R}(F' \otimes_\mathbb{Z} \mathbb{R}) \otimes_\mathbb{R} \det_\mathbb{R}(F' \otimes_\mathbb{Z} \mathbb{R})^{-1} \xrightarrow{\cong} \mathbb{R}$$

where the last arrow is the canonical pairing.

Therefore, in our situation, the image of

$$\det_\mathbb{Z} H^0_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \otimes_\mathbb{Z} \det_\mathbb{Z} \operatorname{Hom}(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z})^{-1}$$

is $D\mathbb{Z} \subset \mathbb{R}$, where $D$ is the determinant of $Reg^\vee_{X,n}$ considered with respect to bases induced by $\mathbb{Z}$-bases of $H^0_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))$ and $\operatorname{Hom}(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z})$. Dually, $D = R_{X,n}$. □

## 6 The special value formula

Now we write down explicitly the special value conjecture $\mathbf{C}(X,n)$ stated in [Bes2021, §4]. For this one considers the canonical isomorphism

$$\lambda\colon \mathbb{R} \xrightarrow{\cong} \bigotimes_{i\in\mathbb{Z}} (\det_{\mathbb{R}} H^i_{W,c}(X,\mathbb{R}(n)))^{(-1)^i} \xrightarrow{\cong} \Big(\bigotimes_{i\in\mathbb{Z}} (\det_{\mathbb{Z}} H^i_{W,c}(X,\mathbb{Z}(n)))^{(-1)^i}\Big) \otimes_{\mathbb{Z}} \mathbb{R}$$
$$\xrightarrow{\cong} (\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n))) \otimes_{\mathbb{Z}} \mathbb{R},$$

where the first isomorphism $\mathbb{R} \cong \bigotimes_{i\in\mathbb{Z}} (\det_{\mathbb{R}} H^i_{W,c}(X,\mathbb{R}(n)))^{(-1)^i}$ comes from the regulator, as will be explained below.

In our case, we are interested in the determinant of Weil-étale complex

$$\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)) \cong \bigotimes_{i\in\mathbb{Z}} \det_{\mathbb{Z}} H^i_{W,c}(X,\mathbb{Z}(n))^{(-1)^i}$$
$$= \det_{\mathbb{Z}} H^1_{W,c}(X,\mathbb{Z}(n))^{-1} \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} H^2_{W,c}(X,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} H^3_{W,c}(X,\mathbb{Z}(n))^{-1}.$$

Using calculations from proposition 4.1, we have

$$\begin{aligned}
&\det_{\mathbb{Z}} H^1_{W,c}(X,\mathbb{Z}(n)) \cong \det_{\mathbb{Z}} H^0_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} T_1,\\
&\det_{\mathbb{Z}} H^2_{W,c}(X,\mathbb{Z}(n)) \cong \det_{\mathbb{Z}} H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))^* \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} H^0(X_{\acute{e}t},\mathbb{Z}^c(n))^D,\\
&\det_{\mathbb{Z}} H^3_{W,c}(X,\mathbb{Z}(n)) \cong \det_{\mathbb{Z}} (H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))_{tors})^D.
\end{aligned}$$

Therefore, we have an isomorphism (up to sign $\pm 1$, after rearranging the terms)

$$\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)) \cong$$
$$= \det_{\mathbb{Z}} H^0_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n))^{-1} \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))^* \otimes_{\mathbb{Z}}$$
$$\det_{\mathbb{Z}}(T_1)^{-1} \det_{\mathbb{Z}} \det_{\mathbb{Z}} H^0(X_{\acute{e}t},\mathbb{Z}^c(n))^D \otimes_{\mathbb{Z}} \det_{\mathbb{Z}}((H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))_{tors})^D)^{-1}.$$

Recall that $T_1$, $H^0(X_{\acute{e}t},\mathbb{Z}^c(n))^D$, $(H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))_{tors})^D$ are finite groups, while the groups $H^0_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n))$ and $H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))^*$ are free of rank $d_n$. Now we consider the canonical trivialization

$$(\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n))) \otimes_{\mathbb{Z}} \mathbb{R} \cong \bigotimes_{i\in\mathbb{Z}} \det_{\mathbb{R}}(H^i_{W,c}(X,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{R}) \cong \mathbb{R}$$

via the regulator morphism

$$\begin{array}{ccc}
H^0_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \otimes \mathbb{R} & & \operatorname{Hom}(H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Z}) \otimes \mathbb{R} \\
\downarrow\cong & & \downarrow\cong \\
H^0_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{R}(n)) & \xrightarrow[\cong]{Reg^\vee_{X,n}} & \operatorname{Hom}(H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R})
\end{array}$$

**6.1. Proposition.** *Under the above trivialization,* $\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)) \subset \mathbb{R}$ *corresponds to* $\alpha^{-1}\,\mathbb{Z} \subset \mathbb{R}$*, where*

$$\begin{aligned}
\alpha &= \frac{|H^0(X_{\acute{e}t},\mathbb{Z}^c(n))^D|}{|T_1|\cdot|(H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))_{tors})^D|}\,R_{X,n}\\
&= 2^\delta\,\frac{|H^0(X_{\acute{e}t},\mathbb{Z}^c(n))|}{|H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))_{tors}|\cdot|H^1(X_{\acute{e}t},\mathbb{Z}^c(n))|}\,R_{X,n},
\end{aligned}$$

*the number* $\delta$ *is given by* (1.5)*, and* $R_{X,n}$ *is the regulator from definition 5.5.*

*Proof.* For finite groups $T_1$, $H^0(X_{ét}, \mathbb{Z}^c(n))^D$, $(H^{-1}(X_{ét}, \mathbb{Z}^c(n))_{tors})^D$, this is [Bes2021, Lemma A.6]. On the other hand, for free groups $H^0_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))$ and $H^{-1}(X_{ét}, \mathbb{Z}^c(n))^*$, this is proposition 5.7 (the corresponding groups sit in degrees 1 and 2 in this case, so the determinant gets inverted). □

We recall that conjecture $\mathbf{C}(X,n)$ from [Bes2021, §4] asserts that $\det_\mathbb{Z} R\Gamma_{W,c}(X, \mathbb{Z}(n)) \subset \mathbb{R}$ corresponds to $\zeta^*(X,n)^{-1}\,\mathbb{Z} \subset \mathbb{R}$. This leads to the following conclusion.

**6.2. Proposition.** *Let $X$ be a one-dimensional arithmetic scheme and $n < 0$. Then the special value conjecture $\mathbf{C}(X,n)$ stated in [Bes2021] is equivalent to the formula (1.4).*

It is proved in [Bes2021, §7] that $\mathbf{C}(X,n)$ holds unconditionally for an abelian one-dimensional arithmetic scheme $X$. Together with the above proposition, this proves theorem 1.2 stated in the introduction.

# 7 Direct proof of the formula

The derivation of the special value formula from our Weil-étale machinery might look rather contrived. In this section we explain how to prove it directly, combining via localization the known special value formulas for $X = \operatorname{Spec} \mathcal{O}_F$ and curves over finite fields $X/\mathbb{F}_q$. This will establish our result, but sections §4 and §6 also serve their purpose: these show how we came up with the formula (1.4) in the first place.

**7.1. Lemma.** *Let $n < 0$.*

*0) If $X$ is a zero-dimensional arithmetic scheme, then*

$$\zeta(X,n) = \pm \frac{1}{|H^1(X_{ét}, \mathbb{Z}^c(n))|}.$$

*1) If $X/\mathbb{F}_q$ be a curve over a finite field, then*

$$\zeta(X,n) = \pm \frac{|H^0(X_{ét}, \mathbb{Z}^c(n))|}{|H^{-1}(X_{ét}, \mathbb{Z}^c(n))| \cdot |H^1(X_{ét}, \mathbb{Z}^c(n))|}.$$

*2) If $X = \operatorname{Spec} \mathcal{O}_F$ for an abelian number field $F/\mathbb{Q}$, then*

$$\zeta(X,n) = \pm \frac{|H^0(X_{ét}, \mathbb{Z}^c(n))|}{|H^{-1}(X_{ét}, \mathbb{Z}^c(n))|}\, R_{X,n}.$$

*In particular, the formula (1.4) holds in these cases.*

*Proof.* In part 0), the motivic cohomology and zeta function do not distinguish between $X$ and $X_{red}$, so that we may suppose that $X$ is a finite disjoint union of $\operatorname{Spec} \mathbb{F}_{q_i}$. Thanks to (3.4),

$$\zeta(X,n) = \prod_i \frac{1}{1 - q_i^{-n}} = \pm \prod_i \frac{1}{|H^1(X_{i,ét}, \mathbb{Z}^c(n))|} = \pm \frac{1}{|H^1(X_{ét}, \mathbb{Z}^c(n))|}.$$

Note that this is the formula (1.4), since $\delta = 0$ in this case, and $H^{-1}(X_{ét}, \mathbb{Z}^c(n)) = H^0(X_{ét}, \mathbb{Z}^c(n)) = 0$ by (3.4).

For part 1), we refer to [Bes2021, §5]. Part 2) follows from [FM2018, §5.8.3], and in particular [FM2018, Proposition 5.35][*]. The formula is equivalent to (1.4), since $2^\delta = |H^1(X_{ét}, \mathbb{Z}(n))|$ by (3.2). □

[*] The latter uses reduction to the Tamagawa number conjecture.

**7.2. Remark.** The special value at $s = 0$ is not necessarily a rational number:

$$\zeta^*(\operatorname{Spec}\mathbb{F}_q, 0) = \lim_{s\to 0} \frac{s}{1 - q^{-s}} = \frac{1}{\log q}.$$

Moreover,

$$H^i(\operatorname{Spec}\mathbb{F}_{q,ét}, \mathbb{Z}^c(0)) = \begin{cases} \mathbb{Z}, & i = 1, \\ \mathbb{Q}/\mathbb{Z}, & i = 2, \\ 0, & i \neq 1, 2. \end{cases}$$

This toy example already shows that it is important that we focus on the case of $n < 0$.

**7.3. Lemma.** *Let $X$ be a one-dimensional arithmetic scheme and let $U \subset X$ be a dense open subset. Then the special value formula (1.4) for $X$ is equivalent to the corresponding formula for $U$.*

*Proof.* Let $Z = X \setminus U$ be the zero-dimensional complement. We have

$$\zeta(X, n) = \zeta(U, n)\,\zeta(Z, n),$$

where

$$\zeta(X, n) \stackrel{?}{=} \pm 2^\delta \frac{|H^0(X_{ét}, \mathbb{Z}^c(n)|}{|H^{-1}(X_{ét}, \mathbb{Z}^c(n))_{tors}| \cdot |H^1(X_{ét}, \mathbb{Z}^c(n))|} R, \tag{7.1}$$

$$\zeta(U, n) \stackrel{?}{=} \pm 2^\delta \frac{|H^0(U_{ét}, \mathbb{Z}^c(n)|}{|H^{-1}(U_{ét}, \mathbb{Z}^c(n))_{tors}| \cdot |H^1(U_{ét}, \mathbb{Z}^c(n))|} R, \tag{7.2}$$

$$\zeta(Z, n) = \pm \frac{1}{|H^1(Z_{ét}, \mathbb{Z}^c(n))|}.$$

Here $\delta = \delta_{X,n} = \delta_{U,n}$, and $R = R_{X,n} = R_{U,n}$ (see lemma 5.6). We note that $|H^{-1}(X_{ét}, \mathbb{Z}^c(n))_{tors}| = |H^{-1}(U_{ét}, \mathbb{Z}^c(n))_{tors}|$. On the other hand, the exact sequence of finite groups

$$0 \to H^0(X_{ét}, \mathbb{Z}^c(n)) \to H^0(U_{ét}, \mathbb{Z}^c(n)) \to H^1(Z_{ét}, \mathbb{Z}^c(n)) \to H^1(X_{ét}, \mathbb{Z}^c(n)) \to H^1(U_{ét}, \mathbb{Z}^c(n)) \to 0$$

gives

$$\frac{|H^0(X_{ét}, \mathbb{Z}^c(n))|}{|H^1(X_{ét}, \mathbb{Z}^c(n))|} = \frac{|H^0(U_{ét}, \mathbb{Z}^c(n))|}{|H^1(U_{ét}, \mathbb{Z}^c(n))|} \cdot \frac{1}{|H^1(Z_{ét}, \mathbb{Z}^c(n))|}.$$

From this we see that (7.1) and (7.2) are equivalent. □

Now lemmas 7.1 and 7.3 above, together with dévissage lemma 2.1 prove theorem 1.2 stated in the introduction.

**7.4. Remark.** Note that $\zeta(\operatorname{Spec}\mathbb{F}_q, n) = \frac{1}{1-q^{-n}} < 0$, and therefore removing $m$ points from $X$ changes the sign of $\zeta^*(X, n)$ by $(-1)^m$. It is not difficult to figure out the sign in any specific example; however, it is not so clear in which terms one may write the general expression for the sign.

# 8 A couple of examples

We finish with two examples that illustrate how the localization arguments work. The first will be rather generic, and consists in specifying the previous section to the case of a nonmaximal order in a number field.

**8.1. Example.** Let $\mathcal{O} \subset \mathcal{O}_F$ be a nonmaximal order in a number field $F/\mathbb{Q}$. Denote $X = \operatorname{Spec}\mathcal{O}$ and $X' = \operatorname{Spec}\mathcal{O}_F$. Geometrically, $\nu\colon X' \to X$ is the normalization. There exist open dense subsets $U \subset X$ and

$U' \subset X'$ such that $\nu$ induces an isomorphism $U' \cong U$. If we denote the corresponding closed complements by $Z = X \setminus U$ and $Z' = X' \setminus U'$, then we have

$$\zeta_{\mathcal{O}}(s) = \frac{\zeta(Z,s)}{\zeta(Z',s)}\,\zeta_F(s).$$

For this identity formulated in classical terms of algebraic number theory, see for instance [Jen1969]. In particular,

$$\zeta^*_{\mathcal{O}}(n) = \pm \frac{|H^1(Z'_{\acute{e}t}, \mathbb{Z}^c(n))|}{|H^1(Z_{\acute{e}t}, \mathbb{Z}^c(n))|}\,\zeta^*_F(n).$$

Now our special value conjectures for $\zeta^*_{\mathcal{O}}(n)$ and $\zeta^*_F(n)$ take form

$$\zeta^*_{\mathcal{O}}(n) \stackrel{?}{=} \pm 2^\delta \,\frac{|H^0(X_{\acute{e}t}, \mathbb{Z}^c(n)|}{|H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{tors}| \cdot |H^1(X_{\acute{e}t}, \mathbb{Z}^c(n)|}\, R, \tag{8.1}$$

$$\zeta^*_F(n) \stackrel{?}{=} \pm 2^\delta \,\frac{|H^0(X'_{\acute{e}t}, \mathbb{Z}^c(n)|}{|H^{-1}(X'_{\acute{e}t}, \mathbb{Z}^c(n))_{tors}| \cdot |H^1(X'_{\acute{e}t}, \mathbb{Z}^c(n)|}\, R. \tag{8.2}$$

Here $|H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{tors}| = |H^{-1}(X'_{\acute{e}t}, \mathbb{Z}^c(n))_{tors}|$, and the exact sequences of finite groups

$$0 \to H^0(X_{\acute{e}t}, \mathbb{Z}^c(n)) \to H^0(U_{\acute{e}t}, \mathbb{Z}^c(n)) \to H^1(Z_{\acute{e}t}, \mathbb{Z}^c(n)) \to H^1(X_{\acute{e}t}, \mathbb{Z}^c(n)) \to H^1(U_{\acute{e}t}, \mathbb{Z}^c(n)) \to 0$$
$$0 \to H^0(X'_{\acute{e}t}, \mathbb{Z}^c(n)) \to H^0(U'_{\acute{e}t}, \mathbb{Z}^c(n)) \to H^1(Z'_{\acute{e}t}, \mathbb{Z}^c(n)) \to H^1(X'_{\acute{e}t}, \mathbb{Z}^c(n)) \to H^1(U'_{\acute{e}t}, \mathbb{Z}^c(n)) \to 0$$

give us

$$\frac{|H^1(Z'_{\acute{e}t}, \mathbb{Z}^c(n))|}{|H^1(Z_{\acute{e}t}, \mathbb{Z}^c(n))|} = \frac{|H^1(X'_{\acute{e}t}, \mathbb{Z}^c(n))|}{|H^1(X_{\acute{e}t}, \mathbb{Z}^c(n))|} \cdot \frac{|H^0(X_{\acute{e}t}, \mathbb{Z}^c(n))|}{|H^0(X'_{\acute{e}t}, \mathbb{Z}^c(n))|},$$

which implies that the formulas (8.1) and (8.2) are equivalent.

The second example is suggested by [JP2020, §7].

**8.2. Example.** Let $p$ be an odd prime. Consider the affine scheme

$$X = \operatorname{Spec}(\mathbb{Z}[1/2] \times_{\mathbb{F}_p} \mathbb{F}_p[t]) = \operatorname{Spec}\mathbb{Z}[1/2] \coprod_{\operatorname{Spec}\mathbb{F}_p} \mathbb{A}^1_{\mathbb{F}_p},$$

which is obtained from $\operatorname{Spec}\mathbb{Z}[1/p]$ and $\mathbb{A}^1_{\mathbb{F}_p} = \operatorname{Spec}\mathbb{F}_p[t]$ by gluing together the points corresponding to prime ideals $(p) \subset \mathbb{Z}[1/2]$ and $(t) \subset \mathbb{F}_p[t]$:

$$\mathbb{Z}[1/2] \times_{\mathbb{F}_p} \mathbb{F}_p[t] = \{(a,f) \in \mathbb{Z}[1/2] \times \mathbb{F}_p[t] \mid a \equiv f(0) \pmod p\}.$$

If we take odd $n < 0$, then there's no regulator. We consider $n = -3$.

First we recall some calculations of motivic cohomology of $\operatorname{Spec}\mathbb{Z}$. According to [KS2008, Proposition 2.1], there is a short exact sequence

$$0 \to K_6(\mathbb{Z}) \to H^2(\operatorname{Spec}\mathbb{Z}_{\acute{e}t}, \mathbb{Z}(4)) \to \mathbb{Z}/2\mathbb{Z} \to 0$$

and an isomorphism $H^1(\operatorname{Spec}\mathbb{Z}_{\acute{e}t}, \mathbb{Z}(4)) \cong K_7(\mathbb{Z})$. We have $K_6(\mathbb{Z}) = 0$ and $K_7(\mathbb{Z}) \cong \mathbb{Z}/240\mathbb{Z}$; see for instance Weibel's survey [Wei2005]*. We conclude that

$$\begin{aligned} H^{-1}(\operatorname{Spec}\mathbb{Z}_{\acute{e}t}, \mathbb{Z}^c(-3)) &\cong H^1(\operatorname{Spec}\mathbb{Z}_{\acute{e}t}, \mathbb{Z}(4)) \cong \mathbb{Z}/240\mathbb{Z},\\ H^0(\operatorname{Spec}\mathbb{Z}_{\acute{e}t}, \mathbb{Z}^c(-3)) &\cong H^2(\operatorname{Spec}\mathbb{Z}_{\acute{e}t}, \mathbb{Z}(4)) \cong \mathbb{Z}/2\mathbb{Z},\\ H^1(\operatorname{Spec}\mathbb{Z}_{\acute{e}t}, \mathbb{Z}^c(-3)) &\cong H^3(\operatorname{Spec}\mathbb{Z}_{\acute{e}t}, \mathbb{Z}(4)) = 0. \end{aligned}$$

*Strictly speaking, this reasoning is *backward*: as explained in [Wei2005], in fact one calculates motivic cohomology of $\operatorname{Spec}\mathcal{O}_F$ in order to obtain the $K$-theory of $\mathcal{O}_F$, not vice versa. Our explanation just serves to reduce everything to the well-known tables of $K$-groups of $\mathbb{Z}$.

We note that, as expected,

$$\zeta(\operatorname{Spec}\mathbb{Z}, -3) = \zeta(-3) = -\frac{B_4}{4} = \frac{1}{120} = \frac{|H^0(\operatorname{Spec}\mathbb{Z}_{ét}, \mathbb{Z}^c(-3))|}{|H^{-1}(\operatorname{Spec}\mathbb{Z}_{ét}, \mathbb{Z}^c(-3))|}.$$

Localization gives

$$\begin{aligned} H^{-1}(\operatorname{Spec}\mathbb{Z}[1/2]_{ét}, \mathbb{Z}^c(-3)) &\cong H^{-1}(\operatorname{Spec}\mathbb{Z}_{ét}, \mathbb{Z}^c(-3)) \cong \mathbb{Z}/240\mathbb{Z}, \\ H^0(\operatorname{Spec}\mathbb{Z}[1/2]_{ét}, \mathbb{Z}^c(-3)) &\cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z}, \\ H^1(\operatorname{Spec}\mathbb{Z}[1/2]_{ét}, \mathbb{Z}^c(-3)) &= H^1(\operatorname{Spec}\mathbb{Z}_{ét}, \mathbb{Z}^c(-3)) = 0. \end{aligned}$$

Arithmetically, this corresponds to the fact that the zeta function of $\operatorname{Spec}\mathbb{Z}[1/2]$ has the same Euler product as $\zeta(s)$, with the factor $\frac{1}{1-2^{-s}}$ removed. Therefore, the value at $s=-3$ should be corrected by $2^3-1=7$.

Now for $\mathbb{A}^1_{\mathbb{F}_p}$, we have

$$H^i(\mathbb{A}^1_{\mathbb{F}_p,ét}, \mathbb{Z}^c(n)) \cong H^{i+2}(\operatorname{Spec}\mathbb{F}_{p,ét}, \mathbb{Z}^c(n-1)) \cong \begin{cases} \mathbb{Z}/(p^{1-n}-1)\mathbb{Z}, & i=-1, \\ 0, & i\neq -1. \end{cases}$$

In particular, the motivic cohomology of $\mathbb{A}^1_{\mathbb{F}_p}$ is concentrated in

$$H^{-1}(\mathbb{A}^1_{\mathbb{F}_p,ét}, \mathbb{Z}^c(-3)) \cong \mathbb{Z}/(p^4-1)\mathbb{Z}.$$

Consider the normalization of $X$, which is given by $X' = \operatorname{Spec}\mathbb{Z}[1/2] \sqcup \mathbb{A}^1_{\mathbb{F}_p}$:

$$\begin{array}{ccc} Z' & \hookrightarrow & X' \\ \downarrow & & \downarrow \\ Z & \hookrightarrow & X \end{array}$$

Here $Z = \{\mathfrak{p}\}$, $Z' = \{\mathfrak{P}, \mathfrak{P}'\}$, and

$$\begin{aligned} \mathfrak{p} &:= \{(a,f) \in \mathbb{Z}[1/2] \times \mathbb{F}_p[t] \mid a \equiv f(0) \equiv 0 \pmod p\}, \\ \mathfrak{P} &:= \{(a,f) \in \mathbb{Z}[1/2] \times \mathbb{F}_p[t] \mid a \equiv 0 \pmod p\}, \\ \mathfrak{P}' &:= \{(a,f) \in \mathbb{Z}[1/2] \times \mathbb{F}_p[t] \mid f(0) \equiv 0 \pmod p\}. \end{aligned}$$

The canonical morphism $X' \to X$ induces an isomorphism

$$X' \setminus Z' \cong X \setminus Z \cong (\operatorname{Spec}\mathbb{Z} \setminus \{(2),(p)\}) \sqcup (\operatorname{Spec}\mathbb{F}_p[t] \setminus (t)).$$

We may calculate via localizations that

$$\begin{aligned} H^{-1}(X_{ét}, \mathbb{Z}^c(-3)) &\cong H^{-1}((X\setminus Z)_{ét}, \mathbb{Z}^c(-3)) \cong \mathbb{Z}/240\mathbb{Z} \oplus \mathbb{Z}/(p^4-1)\mathbb{Z}, \\ H^0(X_{ét}, \mathbb{Z}^c(-3)) &\cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/(p^3-1)\mathbb{Z}, \\ H^1(X_{ét}, \mathbb{Z}^c(-3)) &= 0. \end{aligned}$$

Consequently,

$$\frac{|H^0(X_{ét}, \mathbb{Z}^c(-3))|}{|H^{-1}(X_{ét}, \mathbb{Z}^c(-3))| \cdot |H^1(X_{ét}, \mathbb{Z}^c(-3))|} = \frac{7}{120}\,\frac{p^3-1}{p^4-1}.$$

On the level of zeta-functions,

$$\zeta(X,s) = \zeta(Z,s)\,\zeta(X\setminus Z,s) = \frac{\zeta(Z,s)}{\zeta(Z',s)}\,\zeta(X',s) = \frac{1}{\zeta(\operatorname{Spec}\mathbb{F}_p,s)}\,\zeta(\operatorname{Spec}\mathbb{Z}[1/2],s)\,\zeta(\mathbb{A}^1_{\mathbb{F}_p},s) =$$
$$(1-p^{-s})\,(1-2^{-s})\,\zeta(s)\,\frac{1}{1-p^{1-s}}.$$

In particular, substituting $s=-3$, we obtain $\zeta(X,-3) = -\frac{7}{120}\,\frac{p^3-1}{p^4-1}$.

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