Weil-étale cohomology for arbitrary arithmetic schemes and $n < 0$. Part I: Construction of Weil-étale complexes

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Abstract

Following the ideas of Flach and Morin [FM2018], we define Weil-étale cohomology $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ for an arbitrary scheme X that is separated and of finite type over $\operatorname{Spec} \mathbb{Z}$, and $n < 0$.

Contents

1 Introduction

To a scheme X of finite type over $\operatorname{Spec} \mathbb{Z}$ one can attach its **zeta function**

$$\zeta(X, s) = \prod_{\substack{x \in X \\ \text{closed pt.}}} \frac{1}{1 - \# \kappa(x)^{-s}}$$

(see e.g. [Ser1965]). Lichtenbaum envisioned a cohomology theory, known as **Weil-étale cohomology**, that captures information about the special value of $\zeta(X, s)$ at $s = n$, namely the vanishing order and corresponding residue [Lic2005, Lic2009b, Lic2009a]. For varieties over finite fields $X_{/\mathbb{F}_p}$, it was further studied by Geisser [Gei2004, Gei2006]. Morin gave in [Mor2014] a construction for $X \to \operatorname{Spec} \mathbb{Z}$ separated, of finite type, proper, regular, and $n = 0$. This construction was further generalized by Flach and Morin in [FM2018] to any $n \in \mathbb{Z}$, under the same assumptions on X.

The goal of this work is to remove the assumption that X is proper and regular, and following the ideas from [FM2018], construct Weil-étale cohomology for any X separated and of finite type over $\operatorname{Spec}\mathbb{Z}$ for the case of $n < 0$.

As Flach and Morin already suggest in [FM2018, Remark 3.11], we rework all their constructions in terms of Geisser's dualizing cycle complexes $\mathbb{Z}^c(n)$.

For the reader's convenience, this work is split into two parts. The present Part I is devoted to the construction of Weil-étale complexes $R\Gamma_{W,c}(X, \mathbb{Z}(n))$. Their conjectural relation to the special value of $\zeta(X, s)$ at $s = n$ will be treated in the forthcoming Part II.

Notation and conventions

Complexes. All the constructions will take place in the derived category of abelian groups $\mathbf{D}(\mathbb{Z})$. For our needs, we introduce the following terminology. First recall that a complex of abelian groups $A^\bullet$ is **perfect** if it is bounded (i.e. $H^i(A^\bullet) = 0$ for $|i| \gg 0$), and $H^i(A^\bullet)$ are finitely generated abelian groups.

Definition 1.1. A complex of abelian groups $A^\bullet$ is **almost perfect** if the cohomology groups $H^i(A^\bullet)$ are finitely generated, and bounded, except for possible finite 2-torsion in arbitrarily high degree. That is, $H^i(A^\bullet) = 0$ for $i \ll 0$ and $H^i(A^\bullet)$ is finite 2-torsion for $i \gg 0$.

An abelian group A is of **cofinite type** if it is $\mathbb{Q}/\mathbb{Z}$-dual to a finitely generated abelian group.

A complex of abelian groups $A^\bullet$ is of **cofinite type** if the cohomology groups $H^i(A^\bullet)$ are of cofinite type and bounded.

A complex of abelian groups $A^\bullet$ is **almost of cofinite type** if the cohomology groups $H^i(A^\bullet)$ are of cofinite type and bounded, except for possible finite 2-torsion in arbitrarily high degree.

This terminology is ad hoc and was invented for this text, as such complexes will appear frequently. Some basic observations about almost perfect and almost cofinite type complexes are collected in Appendix A. We note that this finite 2-torsion in arbitrarily high degrees could be removed by working with Artin–Verdier topology $\overline{X}_{\acute{e}t}$ instead of the usual étale topology $X_{\acute{e}t}$; see [FM2018, Appendix A] for more details. We will not use Artin–Verdier topology to simplify the exposition, at the cost of some technical hurdles with 2-torsion.

Étale cohomology. For an arithmetic scheme X and a complex of étale sheaves $\mathcal{F}^\bullet$, we denote by

$$R\Gamma(X_{\acute{e}t}, \mathcal{F}^\bullet) \text{ (resp. } R\Gamma_c(X_{\acute{e}t}, \mathcal{F}^\bullet),\ R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathcal{F}^\bullet))$$

the complex that calculates the corresponding cohomology, resp. cohomology with compact support, and modified cohomology with compact support. For convenience of the reader, the definitions are reviewed in Appendix B. The purpose of $R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathcal{F}^\bullet)$ is to take care of real places of X. There is a canonical projection $R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathcal{F}^\bullet) \to R\Gamma_c(X_{\acute{e}t}, \mathcal{F}^\bullet)$, which is an isomorphism whenever $X(\mathbb{R}) = \varnothing$.

Equivariant cohomology. We denote by $X(\mathbb{C})$ the set of complex points of X equipped with the analytic topology. It carries the natural action of the Galois group $G_\mathbb{R} := \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. For a subring $A \subseteq \mathbb{R}$ we denote by $A(n)$ the constant $G_\mathbb{R}$-equivariant sheaf $(2\pi i)^n A$ on $X(\mathbb{C})$. In what follows, we will be interested in $A = \mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$. For $A = \mathbb{Q}/\mathbb{Z}$ we also consider $\mathbb{Q}/\mathbb{Z}(n) = \mathbb{Q}(n)/\mathbb{Z}(n)$.

In general, a G-equivariant sheaf $\mathcal{F}$ on $X(\mathbb{C})$ can be defined as an espace étalé $\pi\colon E \to X(\mathbb{C})$ with a G-action on E such that the projection π is G-equivariant. The equivariant global sections are defined by $\Gamma(G, X, \mathcal{F}) = \mathcal{F}(X)^G$, with G acting on $\mathcal{F}(X) = \{s\colon X \to E \mid \pi \circ s = id_X\}$ via $(g \cdot s)(x) = g \cdot s(g^{-1} \cdot x)$. Then by definition, the equivariant cohomology is given by the right derived functors of $\Gamma(G, X, -)$. More details on G-equivariant sheaves can be found in [Mor2008, Chapitre 2]. For our modest purposes, it is enough to know that any G-module A gives rise to the corresponding abelian G-equivariant constant sheaf. The latter corresponds to the espace étalé $X(\mathbb{C}) \times A \to X(\mathbb{C})$, with A equipped with the discrete topology.

There is also a complex of sheaves $\mathbb{Z}(n)$ on $X_{\acute{e}t}$, defined below in 1.2. It is not the same as the sheaf $\mathbb{Z}(n) = (2\pi i)^n\,\mathbb{Z}$ on $X(\mathbb{C})$, but there is no possible confusion, since these two live in very different topologies. The notation is deliberate, as we will actually see that the comparison between the étale topology on X and analytic topology on $X(\mathbb{C})$ relates them (see Proposition 6.1).

Assumptions

For the purposes of this article, we will call an **arithmetic scheme** an arbitrary scheme X that is separated and of finite type over $\operatorname{Spec}\mathbb{Z}$. By n we will always denote a strictly negative integer.

Our construction is based on motivic cohomology, defined in terms of Geisser's dualizing cycle complexes $\mathbb{Z}^c(n)$, as introduced and studied in [Gei2010]. Given X and n as above, we state the following conjecture.

Conjecture. $\mathbf{L}^c(X_{ét}, n)$: the cohomology groups $H^i(X_{ét}, \mathbb{Z}^c(n))$ are finitely generated for all $i \in \mathbb{Z}$.

Here "L" stays for "Lichtenbaum". This is analogous to the conjecture $\mathbf{L}(X_{ét}, n)$ in [FM2018, §3.2]. Our construction of Weil-étale complexes $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ will assume $\mathbf{L}^c(X_{ét}, n)$. See §8 for the particular cases when the conjecture is known.

Main results

Before outlining the construction of Weil-étale cohomology, we state the main results of this paper that make it possible. One of our principal objects is the following complex of abelian sheaves $\mathbb{Z}(n)$ on $X_{ét}$.

Definition 1.2 ([FM2018, §3.1], [Gei2004, §7])**.** Let X be an arithmetic scheme and $n < 0$. For a prime p, consider the localization $X[1/p]$, and let μ_{p^r} be the sheaf of p^r-th roots of unity on $X[1/p]$. We define the twist of μ_{p^r} by n as

$$\mu_{p^r}^{\otimes n} = \underline{\operatorname{Hom}}_{X[1/p]}(\mu_{p^r}^{\otimes(-n)}, \mathbb{Z}/p^r).$$

Now $\mathbb{Z}(n)$ is the complex of sheaves on $X_{ét}$ given by

$$\mathbb{Z}(n) = \mathbb{Q}/\mathbb{Z}(n)[-1], \quad \text{where } \mathbb{Q}/\mathbb{Z}(n) = \bigoplus_p \varinjlim_r j_{p!}\mu_{p^r}^{\otimes n},$$

and j_p is the canonical open immersion $X[1/p] \to X$.

The above sheaves $\mathbb{Z}(n)$ should not be confused with cycle complexes; the latter are $\mathbb{Z}^c(n)$ in the setting of this paper. In §2 we prove the following arithmetic duality theorem relating the two.

Theorem I. *Assuming the conjecture* $\mathbf{L}^c(X_{ét}, n)$, *there is a quasi-isomorphism of complexes*

$$R\widehat{\Gamma}_c(X_{ét}, \mathbb{Z}(n)) \xrightarrow{\cong} R\operatorname{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]).$$

The second result we will need is related to the following morphism of complexes.

Definition 1.3. We define $v_\infty^*\colon R\Gamma_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))$ as the morphism in the derived category $\mathbf{D}(\mathbb{Z})$ induced by the comparison of étale and analytic topology

$$\Gamma_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n)) \to \Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \alpha^*\mathbb{Q}/\mathbb{Z}(n)) \cong \Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))$$

(see Proposition B.3 and 6.1). Then we let $u_\infty^*\colon R\Gamma_c(X_{ét}, \mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))$ be the composition

$$R\Gamma_c(X_{ét}, \mathbb{Z}(n)) := R\Gamma_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n))[-1] \xrightarrow{v_\infty^*[-1]} R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))[-1] \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))$$

where the last arrow is induced by $\mathbb{Q}/\mathbb{Z}(n)[-1] \to \mathbb{Z}(n)$, which comes from the distinguished triangle of constant $G_\mathbb{R}$-equivariant sheaves $\mathbb{Z}(n) \to \mathbb{Q}(n) \to \mathbb{Q}/\mathbb{Z}(n) \to \mathbb{Z}(n)[1]$.

Then §6 is devoted to the following result.

Theorem II. *The morphism* u_∞^* *is torsion, i.e. it is a torsion element in the abelian group*

$$\operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(R\Gamma_c(X_{ét}, \mathbb{Z}(n)), R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))).$$

Outline of the construction of Weil-étale cohomology

Here we outline the structure of this paper, as well as our construction of Weil-étale complexes $R\Gamma_{W,c}(X, \mathbb{Z}(n))$.

First, §2 is devoted to a proof of Theorem I. Some of its consequences are deduced in §4. Namely, if we assume the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, then $R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n))$ is an almost perfect complex, while $R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n))$ is almost of cofinite type in the sense of Definition 1.1. For this we first make a little digression in §3 to analyze what kind of complexes we obtain for $G_{\mathbb{R}}$-equivariant cohomology on $X(\mathbb{C})$.

Theorem I is used in §5 to define a morphism $\alpha_{X,n}$ in the derived category (see definition 5.1), and declare $R\Gamma_{fg}(X, \mathbb{Z}(n))$ to be its cone:

$$R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \xrightarrow{\alpha_{X,n}} R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \to R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1])$$

The notation *fg* comes from the fact that the complex $R\Gamma_{fg}(X, \mathbb{Z}(n))$ is almost perfect in the sense of Definition 1.1. Thanks to specific properties of the involved complexes, it turns out that $R\Gamma_{fg}(X, \mathbb{Z}(n))$ is defined up to a *unique* isomorphism in the derived category (something one usually does not expect from a cone).

Then in §6 we establish Theorem II, and it is used in §7 to define Weil-étale complexes $R\Gamma_{W,c}(X, \mathbb{Z}(n))$. For this we deduce from Theorem II that $u_\infty^* \circ \alpha_{X,n} = 0$, which implies that there exists a morphism in the derived category

$$i_\infty^* \colon R\Gamma_{fg}(X, \mathbb{Z}(n)) \to R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))$$

(see the diagram below). We pick a mapping fiber of i_∞^* and call it $R\Gamma_{W,c}(X, \mathbb{Z}(n))$, which turns out to be a perfect complex. Finally, in §8 we consider the cases of X for which the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$ is known, and hence our results hold unconditionally.

There are two appendices to this paper: Appendix A collects some lemmas from homological algebra, and Appendix B reviews the definitions of étale cohomology with compact support $R\Gamma_c(X_{\acute{e}t}, -)$ and its modified version $R\widehat{\Gamma}_c(X_{\acute{e}t}, -)$.

The definition of $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ fits in the following commutative diagram involving distinguished triangles in the derived category $\mathbf{D}(\mathbb{Z})$:

$$\begin{array}{ccccccc}
 & & & & R\Gamma_{W,c}(X, \mathbb{Z}(n)) & & \\
 & & & & \downarrow & & \\
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & \xrightarrow[\text{dfn. } 5.1]{\alpha_{X,n}} & R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X, \mathbb{Z}(n)) & \longrightarrow & \cdots \\
\downarrow & & \downarrow{\scriptstyle u_\infty^*} & & \downarrow{\scriptstyle i_\infty^*} & & \downarrow \\
0 & \longrightarrow & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) & \xrightarrow{\text{id}} & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) & \longrightarrow & 0 \\
 & & & & \downarrow & & \\
 & & & & R\Gamma_{W,c}(X, \mathbb{Z}(n))[1] & &
\end{array}$$

Our construction follows [FM2018], and in particular, the resulting complex is the same whenever X is proper and regular, which is the assumption considered by Flach and Morin. Here is a brief comparison between the notation.

this paper	**Flach–Morin**
Geisser's dualizing cycle complexes	Bloch's cycle complexes
$\mathbb{Z}^c(n)$	$\mathbb{Z}(d-n)[2d]$, $d = \dim X$
$R\Gamma_{fg}(X, \mathbb{Z}(n))$	$R\Gamma_W(\overline{X}, \mathbb{Z}(n))$, up to finite 2-torsion for $i \gg 0$
$R\Gamma_{W,c}(X, \mathbb{Z}(n))$	$R\Gamma_{W,c}(X, \mathbb{Z}(n))$

Further properties of $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ that are needed to establish its conjectural relation to the special value of $\zeta(X, s)$ at $s = n$ will be treated in the second part of this article.

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2 Proof of Theorem I

At the heart of our constructions is a certain arithmetic duality theorem for cycle complexes obtained by Thomas Geisser in [Gei2010]. The goal of this section is to deduce Theorem I from Geisser's duality. We would like to establish a quasi-isomorphism of complexes

$$R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n)) \xrightarrow{\cong} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]).$$

Here $R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n))$ denotes the modified étale cohomology with compact support, which is reviewed in the appendix B. We note that [Gei2010] uses the notation "$R\Gamma_c$" for our "$R\widehat{\Gamma}_c$", but we take extra care to distinguish the two things, as we will also need the usual étale cohomology with compact support $R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n))$.

We split our proof of Theorem I in two propositions.

Proposition 2.1. *For any $n < 0$ we have a quasi-isomorphism of complexes*

$$R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n)) \cong \varinjlim_m R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}/m\mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]). \tag{2.1}$$

Proof. We unwind our definition of $\mathbb{Z}(n)$ for $n < 0$ and reduce everything to the results from [Gei2010].

As we have $\mathbb{Z}(n) := \bigoplus_p \varinjlim_r j_{p!}\mu_{p^r}^{\otimes n}[-1]$, it will be enough to show that for every prime p and $r = 1, 2, 3, \ldots$ there is a quasi-isomorphism of complexes

$$R\widehat{\Gamma}_c(X_{\acute{e}t}, j_{p!}\mu_{p^r}^{\otimes n}[-1]) \cong R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c/p^r(n)), \mathbb{Q}/\mathbb{Z}[-2]),$$

and then pass to the corresponding filtered colimits.

As in the Definition 1.2, here j_p denotes the canonical open immersion $j_p \colon X[1/p] \hookrightarrow X$. We further denote by f the structure morphism $X \to \operatorname{Spec}\mathbb{Z}$ and by f_p the structure morphism $X[1/p] \to \operatorname{Spec}\mathbb{Z}[1/p]$:

$$\begin{array}{ccc} X[1/p] & \xhookrightarrow{j_p} & X \\ {\scriptstyle f_p}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec}\mathbb{Z}[1/p] & \hookrightarrow & \operatorname{Spec}\mathbb{Z} \end{array}$$

As we are going to change the base scheme, let us write $\mathrm{Hom}_X(-,-)$ for the Hom between sheaves on $X_{\acute{e}t}$ and $\underline{\mathrm{Hom}}_X(-,-)$ for the internal Hom. Instead of $\mathrm{Hom}_{\operatorname{Spec} R}$, we will simply write Hom_R.

By [Gei2010, Proposition 7.10, (c)], we have the following exchange formulas. If we work with complexes of constructible sheaves on the étale site of schemes over the spectrum of a number ring $\operatorname{Spec}\mathcal{O}$, then for a morphism ϕ of such schemes we have

$$R\phi_*\mathcal{D}(\mathcal{F}) \cong \mathcal{D}(R\phi_!\mathcal{F}), \tag{2.2}$$

$$R\phi^!\mathcal{D}(\mathcal{G}) \cong \mathcal{D}(\phi^*\mathcal{G}), \tag{2.3}$$

where the dualization is given by

$$\mathcal{D}(\mathcal{F}^\bullet) := R\underline{\mathrm{Hom}}_X(\mathcal{F}^\bullet, \mathbb{Z}^c(0)).$$

Applying the exchange formula (2.2) to our situation, we get

$$R\underline{\mathrm{Hom}}_X(j_{p!}\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}_X^c(0)) \cong Rj_{p*}R\underline{\mathrm{Hom}}_{X[1/p]}(\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}_{X[1/p]}^c(0)). \tag{2.4}$$

Using the other exchange formula (2.3), we may identify the sheaf $R\underline{\mathrm{Hom}}_{X[1/p]}(\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}_{X[1/p]}^c(0))$:

$$\begin{aligned} R\underline{\mathrm{Hom}}_{X[1/p]}(\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}_{X[1/p]}^c(0)) &\cong R\underline{\mathrm{Hom}}_{X[1/p]}(f_p^*\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}_{X[1/p]}^c(0)) && (2.5)\\ &\cong Rf_p^!R\underline{\mathrm{Hom}}_{\mathbb{Z}[1/p]}(\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}_{\mathbb{Z}[1/p]}^c(0)) && (2.6)\\ &\cong Rf_p^!R\underline{\mathrm{Hom}}_{\mathbb{Z}[1/p]}(\mu_{p^r}^{\otimes n}[-1], \mathbb{G}_{\mathrm{m}}[1]) && (2.7)\\ &\cong Rf_p^!R\underline{\mathrm{Hom}}_{\mathbb{Z}[1/p]}(\mu_{p^r}^{\otimes n}, \mathbb{G}_{\mathrm{m}})[2] && (2.8)\\ &\cong Rf_p^!\mu_{p^r}^{\otimes(1-n)}[2] && (2.9) \end{aligned}$$

Here (2.5) simply means that the sheaf $\mu_{p^r}^{\otimes n}$ on $X[1/p]$ is the same as the inverse image of the corresponding sheaf on $\operatorname{Spec}\mathbb{Z}[1/p]$. The quasi-isomorphism (2.6) is the first exchange formula. Then, (2.7) is the fact that the complex $\mathbb{Z}_{\mathbb{Z}[1/p]}^c(0)$ is quasi-isomorphic to $\mathbb{G}_{\mathrm{m}}[1]$ according to [Gei2010, Lemma 7.4]. Thanks to [Gei2004, Theorem 1.2], we may identify the sheaf $\mu_{p^r}^{\otimes(1-n)}$:

$$\mu_{p^r}^{\otimes(1-n)} \cong \mathbb{Z}_{\mathbb{Z}[1/p]}/p^r(1-n) = \mathbb{Z}_{\mathbb{Z}[1/p]}^c/p^r(n)[-2]. \tag{2.10}$$

Then [Gei2010, Corollary 7.9] tells us that

$$Rf_p^!\mathbb{Z}_{\mathbb{Z}[1/p]}^c/p^r(n) \cong \mathbb{Z}_{X[1/p]}^c/p^r(n). \tag{2.11}$$

Finally, thanks to [Gei2010, Theorem 7.2 (a)] and [Gei2010, Proposition 2.3], we have $\mathbb{Z}_{X[1/p]}^c/p^r(n) \cong j_p^*\mathbb{Z}_X^c/p^r(n)$, and all the above gives

$$R\underline{\mathrm{Hom}}_X(j_{p!}\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}_X^c(0)) \cong Rj_{p*}j_p^*\mathbb{Z}_X^c/p^r(n) \cong \mathbb{Z}_X^c/p^r(n). \tag{2.12}$$

After applying $R\Gamma(X_{ét}, -)$, we get a quasi-isomorphism of complexes of abelian groups

$$R\mathrm{Hom}(j_{p!}\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}_X^c(0)) \cong R\Gamma(X_{ét}, \mathbb{Z}_X^c/p^r(n)). \tag{2.13}$$

Now according to the duality theorem [Gei2010, Theorem 7.8], we have

$$R\mathrm{Hom}(j_{p!}\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c(0)) \cong R\mathrm{Hom}(R\widehat{\Gamma}_c(X_{ét}, j_{p!}\mu_{p^r}^{\otimes n}[-1]), \mathbb{Q}/\mathbb{Z}[-2]). \tag{2.14}$$

What we obtain at the end is a quasi-isomorphism

$$R\Gamma(X_{ét}, \mathbb{Z}^c/p^r(n)) \cong R\mathrm{Hom}(R\widehat{\Gamma}_c(X_{ét}, j_{p!}\mu_{p^r}^{\otimes n}[-1]), \mathbb{Q}/\mathbb{Z}[-2]).$$

This is almost what we need: if we apply $R\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z}[-2])$, then, as $\widehat{H}_c^i(X_{ét}, j_{p!}\mu_{p^r}^{\otimes n}[-1])$ are finite groups (because the sheaves $j_{p!}\mu_{p^r}^{\otimes n}$ are constructible), we have

$$\begin{aligned} &R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c/p^r(n)), \mathbb{Q}/\mathbb{Z}[-2]) \cong \\ &\qquad R\mathrm{Hom}(R\mathrm{Hom}(R\widehat{\Gamma}_c(X_{ét}, j_{p!}\mu_{p^r}^{\otimes n}[-1]), \mathbb{Q}/\mathbb{Z}[-2]), \mathbb{Q}/\mathbb{Z}[-2]) \\ &\qquad\qquad \cong R\widehat{\Gamma}_c(X_{ét}, j_{p!}\mu_{p^r}^{\otimes n}[-1]). \quad \square \end{aligned}$$

Now to conclude the proof of Theorem I, we identify the complex on the right hand side of (2.1). For this we will need the conjecture $\mathbf{L}^c(X_{ét}, n)$.

Proposition 2.2. *Assuming the conjecture* $\mathbf{L}^c(X_{ét}, n)$*, there is a quasi-isomorphism of complexes*

$$\varinjlim_m R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}/m\mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]) \cong R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]).$$

Proof. As $\mathbb{Z}^c(n)$ is a complex of flat sheaves, the short exact sequence of abelian groups

$$0 \to \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$$

induces a short exact sequence of sheaves

$$0 \to \mathbb{Z}^c(n) \xrightarrow{\times m} \mathbb{Z}^c(n) \to \mathbb{Z}/m\mathbb{Z}^c(n) \to 0 \tag{2.15}$$

The morphism $\mathbb{Z}^c(n) \to \mathbb{Z}/m\mathbb{Z}^c(n)$ induces certain morphisms in cohomology

$$H^i(X_{ét}, \mathbb{Z}^c(n)) \to H^i(X_{ét}, \mathbb{Z}/m\mathbb{Z}^c(n)).$$

We claim that if we pass to the duals $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ and then to the filtered colimits $\varinjlim_m$, then we obtain an isomorphism. (Note that both $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ and $\varinjlim_m$ are exact.)

The short exact sequence (2.15) induces a long exact sequence in cohomology

$$\cdots \longrightarrow H^i(X_{ét}, \mathbb{Z}^c(n)) \xrightarrow{\times m} H^i(X_{ét}, \mathbb{Z}^c(n)) \longrightarrow H^i(X_{ét}, \mathbb{Z}/m\mathbb{Z}^c(n)) \xrightarrow{\delta^i} H^{i+1}(X_{ét}, \mathbb{Z}^c(n)) \xrightarrow{\times m} H^{i+1}(X_{ét}, \mathbb{Z}^c(n)) \to H^{i+1}(X_{ét}, \mathbb{Z}/m\mathbb{Z}^c(n)) \to \cdots$$

We further have exact sequences

$$H^i(X_{ét}, \mathbb{Z}^c(n)) \xrightarrow{\times m} H^i(X_{ét}, \mathbb{Z}^c(n)) \longrightarrow H^i(X_{ét}, \mathbb{Z}^c(n))_m \longrightarrow 0, \qquad H^i(X_{ét}, \mathbb{Z}^c(n))_m = \ker \delta^i$$

$$0 \to {}_mH^{i+1}(X_{ét}, \mathbb{Z}^c(n)) \to H^{i+1}(X_{ét}, \mathbb{Z}^c(n)) \xrightarrow{\times m} H^{i+1}(X_{ét}, \mathbb{Z}^c(n)), \qquad {}_mH^{i+1}(X_{ét}, \mathbb{Z}^c(n)) = \mathrm{im}\, \delta^i$$

that give us

$$0 \to H^i(X_{ét}, \mathbb{Z}^c(n))_m \to H^i(X_{ét}, \mathbb{Z}/m\mathbb{Z}^c(n)) \to {}_mH^{i+1}(X_{ét}, \mathbb{Z}^c(n)) \to 0$$

Now if we take $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ and filtered colimits $\varinjlim_m$, we get

$$0 \to \varinjlim_m \mathrm{Hom}({}_mH^{i+1}(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \to \varinjlim_m \mathrm{Hom}(H^i(X_{ét}, \mathbb{Z}/m\mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \to \varinjlim_m \mathrm{Hom}(H^i(X_{ét}, \mathbb{Z}^c(n))_m, \mathbb{Q}/\mathbb{Z}) \to 0 \tag{2.16}$$

By the conjecture $\mathbf{L}^c(X_{ét}, n)$, the group $H^{i+1}(X_{ét}, \mathbb{Z}^c(n))$ is finitely generated, and therefore the first $\varinjlim_m$ in the short exact sequence (2.16) vanishes, and we obtain isomorphisms

$$\varinjlim_m \mathrm{Hom}(H^i(X_{ét}, \mathbb{Z}^c(n))_m, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \varinjlim_m \mathrm{Hom}(H^i(X_{ét}, \mathbb{Z}/m\mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}).$$

It remains to note that the first $\varinjlim_m$ above is canonically isomorphic to $\mathrm{Hom}(H^i(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z})$, again, thanks to finite generation of $H^i(X_{ét}, \mathbb{Z}^c(n))$, assuming the conjecture $\mathbf{L}^c(X_{ét}, n)$. ∎

3 $G_{\mathbb{R}}$-equivariant cohomology of $X(\mathbb{C})$

Given an arithmetic scheme X, we consider its complex points $X(\mathbb{C})$ equipped with the usual analytic topology. It carries a natural action of $G_{\mathbb{R}} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. We consider $G_{\mathbb{R}}$-modules

$$\mathbb{Z}(n) := (2\pi i)^n\,\mathbb{Z}, \quad \mathbb{Q}(n) := (2\pi i)^n\,\mathbb{Q}, \quad \mathbb{Q}/\mathbb{Z}(n) := \mathbb{Q}(n)/\mathbb{Z}(n)$$

as constant $G_{\mathbb{R}}$-equivariant sheaves on $X(\mathbb{C})$. Then $R\Gamma_c(X(\mathbb{C}), A(n))$ (the complex that calculates singular cohomology with compact support of $X(\mathbb{C})$ with coefficients in $A(n)$) is a complex of $G_{\mathbb{R}}$-modules, and we may further take group cohomology (resp. Tate cohomology), which leads to complexes

$$\begin{aligned} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), A(n)) &:= R\Gamma_c(G_{\mathbb{R}}, R\Gamma_c(X(\mathbb{C}), A(n))), \\ R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), A(n)) &:= R\widehat{\Gamma}_c(G_{\mathbb{R}}, R\Gamma(X(\mathbb{C}), A(n))). \end{aligned}$$

By definition, this is the $G_{\mathbb{R}}$-equivariant cohomology (resp. Tate cohomology) with compact support of $X(\mathbb{C})$ with coefficients in $A(n)$.

In this section we analyze what kind of complexes we obtain. All subsequent lemmas are summarized in the below table.

complex	**type**
$R\Gamma_c(X(\mathbb{C}), \mathbb{Z}(n))$	perfect
$R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))$	almost perfect
$R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))$	finite 2-torsion cohomology
$R\Gamma_c(X(\mathbb{C}), \mathbb{Q}(n))$	perfect of $\mathbb{Q}$-vector spaces
$R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}(n))$	quasi-isomorphic to 0
$R\Gamma_c(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))$	cofinite type
$R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))$	almost cofinite type
$R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))$	finite 2-torsion cohomology

First we consider the usual, non-equivariant cohomology. As is well-known, $R\Gamma_c(X(\mathbb{C}), \mathbb{Z})$ are perfect complexes, i.e. the cohomology groups $H^i_c(X(\mathbb{C}), \mathbb{Z})$ are finitely generated and zero for $|i| \gg 0$. The same is true for $H^i_c(X(\mathbb{C}), \mathbb{Z}(n))$ (here the twist in $\mathbb{Z}(n)$ only changes the $G_{\mathbb{R}}$-module structure). For $\mathbb{Q}/\mathbb{Z}$ coefficients, the following result will be useful.

Lemma 3.1. *Given an extension of abelian groups $0 \to A \to B \to C \to 0$, if A and C are of cofinite type, then B is of cofinite type as well.*

Proof. For a finitely generated abelian group G, denote $G^D := \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$. We claim that if G' and G'' are finitely generated abelian groups, then every extension

$$0 \to G'^D \to E \to G''^D \to 0$$

is equivalent to the $\mathbb{Q}/\mathbb{Z}$-dual of an extension

$$0 \to G'' \to G \to G' \to 0$$

where G is a finitely generated abelian group. In other words, we want to show that

$$\begin{gathered} E(G', G'') \to E(G''^D, G'^D), \\ [G'' \rightarrowtail G \twoheadrightarrow G'] \mapsto [G'^D \rightarrowtail G^D \twoheadrightarrow G''^D] \end{gathered}$$

is an isomorphism of Yoneda Exts.

For this we first note that $E(G', G'') \cong \mathrm{Ext}^1_{\mathbb{Z}}(G', G'')$ and $E(G''^D, G'^D) \cong \mathrm{Ext}^1_{\mathbb{Z}}(G''^D, G'^D)$ are indeed isomorphic finite groups, e.g. by considering separately the cases $G', G'' = \mathbb{Z}, \mathbb{Z}/m\mathbb{Z}$ and using additivity of Ext. For finitely generated abelian groups, the functor $G \rightsquigarrow (G^D)^D$ is the same as profinite completion $G \rightsquigarrow \widehat{G} = G \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. Therefore, the composition

$$E(G', G'') \xrightarrow{D} E(G''^D, G'^D) \xrightarrow{D} E(\widehat{G'}, \widehat{G''})$$

is an isomorphism. □

Lemma 3.2. *The complex $R\Gamma_c(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))$ is of cofinite type.*

Proof. The statement follows from the distinguished triangle

$$R\Gamma_c(X(\mathbb{C}), \mathbb{Z}) \to R\Gamma_c(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \to R\Gamma_c(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}) \to R\Gamma_c(X(\mathbb{C}), \mathbb{Z})[1]$$

Indeed, the associated long exact sequence in cohomology

$$\cdots \to H^i_c(X(\mathbb{C}), \mathbb{Z}) \to H^i_c(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^i_c(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}) \to \\ H^{i+1}_c(X(\mathbb{C}), \mathbb{Z}) \to H^{i+1}_c(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \to \cdots$$

shows that $H^i_c(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z})$ is an extension of a finite group by a group of cofinite type:

$$0 \to \mathrm{coker}(H^i_c(X(\mathbb{C}), \mathbb{Z}) \to H^i_c(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}) \to \\ H^i_c(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}) \to \\ \ker(H^{i+1}_c(X(\mathbb{C}), \mathbb{Z}) \to H^{i+1}_c(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}) \to 0$$

Then according to Lemma 3.1, $H^i_c(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z})$ is of cofinite type.

Finally, $H^i_c(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z})$ vanishes for $|i| \gg 0$, because $H^i_c(X(\mathbb{C}), \mathbb{Z})$ does. □

Now we turn to $G_{\mathbb{R}}$-equivariant cohomology. In this case we make use of spectral sequences

$$E_2^{pq} = H^p(G_{\mathbb{R}}, H^q_c(X(\mathbb{C}), A(n))) \Longrightarrow H^{p+q}_c(G_{\mathbb{R}}, X(\mathbb{C}), A(n)),$$
$$E_2^{pq} = \widehat{H}^p(G_{\mathbb{R}}, H^q_c(X(\mathbb{C}), A(n))) \Longrightarrow \widehat{H}^{p+q}_c(G_{\mathbb{R}}, X(\mathbb{C}), A(n)).$$

Here $H^q_c(X(\mathbb{C}), A(n)) = 0$ for $q < 0$ and $q \gg 0$. We recall that the cohomology groups $H^p(G_{\mathbb{R}}, H^q_c(X(\mathbb{C}), A(n)))$ are killed by $2 = \#G_{\mathbb{R}}$ for all $p > 0$; for this see e.g. [Wei1994, Theorem 6.5.8]. For Tate cohomology, the same argument shows that $\widehat{H}^p(G_{\mathbb{R}}, H^q_c(X(\mathbb{C}), A(n)))$ are 2-torsion for all p, including $p = 0$.

Lemma 3.3. *For $A = \mathbb{Q}$ we have the following.*

1) $H^p(G_{\mathbb{R}}, H^q_c(X(\mathbb{C}), \mathbb{Q}(n))) = 0$ for all $p > 0$,

2) $\widehat{H}^p(G_{\mathbb{R}}, H^q_c(X(\mathbb{C}), \mathbb{Q}(n))) = 0$ for all p,

3) $\widehat{H}^p(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}(n))) = 0$ for all p.

Proof. The cohomology groups in 1) and 2) are 2-torsion $\mathbb{Q}$-vector spaces, hence trivial. Part 3) follows from the spectral sequence

$$E_2^{pq} = \widehat{H}^p(G_{\mathbb{R}}, H^q_c(X(\mathbb{C}), \mathbb{Q}(n))) \Longrightarrow \widehat{H}^{p+q}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}(n)).$$ □

Lemma 3.4. *We have a quasi-isomorphism of complexes*

$$R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n)[-1]) \cong R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)).$$

Proof. The short exact sequence of $G_{\mathbb{R}}$-equivariant sheaves on $X(\mathbb{C})$

$$0 \to \mathbb{Z}(n) \to \mathbb{Q}(n) \to \mathbb{Q}/\mathbb{Z}(n) \to 0$$

gives a distinguished triangle

$$R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \to R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}(n)) \to R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n)) \to R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))[1]$$

By the previous lemma, here $R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}(n)) \cong 0$. □

Lemma 3.5. *The groups*

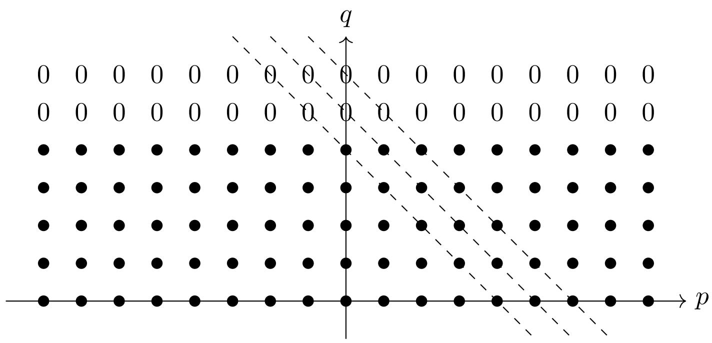
$$\widehat{H}^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \cong \widehat{H}^{i-1}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))$$

are finite 2-torsion.

Proof. As we already recalled, $\widehat{H}^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))$ are 2-torsion. To see that the torsion is finite, consider the spectral sequence

$$E_2^{pq} = \widehat{H}^p(G_{\mathbb{R}}, H^q_c(X(\mathbb{C}), \mathbb{Z}(n))) \Longrightarrow \widehat{H}^{p+q}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)).$$

The groups $H^q_c(X(\mathbb{C}), \mathbb{Z}(n))$ are finitely generated and vanish for $q \gg 0$ and $q < 0$. This means that the second page of the spectral sequence looks like



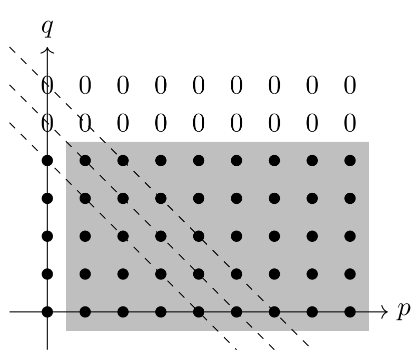
where all objects are *finite* 2-torsion. □

Lemma 3.6. *The complex $R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))$ is almost perfect.*

Proof. Similarly, we consider the spectral sequence

$$E_2^{pq} = H^p(G_{\mathbb{R}}, H^q_c(X(\mathbb{C}), \mathbb{Z}(n))) \Longrightarrow H^{p+q}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)).$$

Here $H^p(G_{\mathbb{R}}, H^q_c(X(\mathbb{C}), \mathbb{Z}(n)))$ is not necessarily 2-torsion for $p = 0$, and the second page looks like



where the shaded part E_2^{pq}, $p > 0$ consists of finitely generated 2-torsion groups, the line E_2^{0q} consists of finitely generated groups, and the objects E_2^{pq} are zero for $q \gg 0$. It follows that the groups $H^i(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))$ are all finitely generated as well, and they are torsion for $i \gg 0$. This is in fact 2-torsion, and we may see this as follows. If $P_\bullet \twoheadrightarrow \mathbb{Z}$ is the bar-resolution of $\mathbb{Z}$ by free $\mathbb{Z}G_\mathbb{R}$-modules, then the morphism of complexes

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & P_3 & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow 0 \\
& \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2-N} & \\
\cdots \longrightarrow & P_3 & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow 0
\end{array}$$

which induces multiplication by 2 on $H^i(G, -)$ for $i > 0$ is null-homotopic [Wei1994, Theorem 6.5.8]. It is not multiplication by 2 in degree 0, but as the complex $R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))$ is bounded, we see that it induces multiplication by 2 on $H^i(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))$ for $i \gg 0$. □

Similarly for $\mathbb{Q}/\mathbb{Z}$-coefficients, we have the following observation.

Lemma 3.7. *The complex $R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))$ is almost of cofinite type.*

Proof. Consider the spectral sequence

$$E_2^{pq} = H^p(G_\mathbb{R}, H_c^q(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))) \Longrightarrow H^{p+q}(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n)).$$

The second page will have groups of cofinite type on the line E_2^{0q} and finite 2-torsion groups E_2^{pq} for $p > 0$. We have filtrations

$$H^{p+q} = F^0(H^{p+q}) \supseteq F^1(H^{p+q}) \supseteq F^2(H^{p+q}) \supseteq \cdots \supseteq F^{p+q}(H^{p+q}) \supset F^{p+q+1}(H^{p+q}) = 0 \tag{3.1}$$

where

$$0 \to F^{p+1}(H^{p+q}) \to F^p(H^{p+q}) \to E_\infty^{pq} \to 0$$

Note that E_∞^{0q} will be groups of cofinite type, and E_∞^{pq} will be finite 2-torsion groups for $p > 0$, as we are going to have

$$0 \to E_{r+1}^{0q} \to E_r^{0q} \to T \to 0$$

where T is finite 2-torsion, and similarly,

$$E_{r+1}^{pq} \cong \ker d_r^{pq} / \operatorname{im} d_r^{p-r,q+r-1}$$

$$E_r^{p-r,q+r-1} \xrightarrow{d_r^{p-r,q+r-1}} E_r^{pq} \xrightarrow{d_r^{pq}} E_r^{p+r,q-r+1}$$

where E_r^{pq} is finite 2-torsion for $p > 0$. It follows by induction that all terms of the filtration (3.1) are finite groups, except for $F^0(H^{p+q}) = H^{p+q}$ itself, which is of cofinite type, being an extension of a group of cofinite type E_∞^{0q} by a finite group $F^1(H^{p+q})$ (see Lemma 3.1). We also see that H^{p+q} is 2-torsion for $p+q \gg 0$. □

4 Some consequences of Theorem I

We note that the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$ actually implies that the complex $R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n))$ is bounded from below.

Lemma 4.1. *Assuming the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, the complex $R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n))$ is bounded from below: one has $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) = 0$ for $i < 2 \dim X$.*

Proof. Since the complex of sheaves $\mathbb{Z}^c(n)$ is flat, the short exact sequence of abelian groups

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

gives us a short exact sequence of étale sheaves

$$0 \to \mathbb{Z}^c(n) \to \mathbb{Q}^c(n) \to \mathbb{Q}/\mathbb{Z}^c(n) \to 0$$

and then applying $R\Gamma(X_{\acute{e}t}, -)$, we obtain a distinguished triangle in $\mathbf{D}(\mathbb{Z})$

$$R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)) \to R\Gamma(X_{\acute{e}t}, \mathbb{Q}^c(n)) \to R\Gamma(X_{\acute{e}t}, \mathbb{Q}/\mathbb{Z}^c(n)) \to R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n))[1]$$

Now according to [Mor2014, Lemma 5.12] (note that the proof there also uses Geisser's duality theorem), we have

$$H^i(X_{\acute{e}t}, \mathbb{Q}/\mathbb{Z}^c(n)) = 0 \quad \text{for } i < -2 \dim X,$$

and the above triangle implies that

$$H^i(X_{\acute{e}t}, \mathbb{Q}^c(n)) \cong H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) \quad \text{for } i < -2 \dim X.$$

However, $H^i(X_{\acute{e}t}, \mathbb{Q}^c(n))$ is a $\mathbb{Q}$-vector space, and according to the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, the groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are finitely generated over $\mathbb{Z}$. This means that for $i < -2 \dim X$ these groups are trivial. □

Lemma 4.2. *The canonical morphism* $\phi^i \colon \widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to H^i_c(X_{\acute{e}t}, \mathbb{Z}(n))$ *sits in a long exact sequence*

$$\cdots \to \widehat{H}^{i-1}_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \to \widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \xrightarrow{\phi^i} H^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to \widehat{H}^i_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \to \cdots$$

where the groups $\widehat{H}^i_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))$ *are finite* 2*-torsion. In particular, the kernel and cokernel of* ϕ^i *is finite* 2*-torsion.*

Proof. The exact sequence follows from the definition of modified étale cohomology with compact support and Artin's comparison theorem. This is proved in [FM2018, Lemma 6.14]. The fact that $\widehat{H}^i_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))$ are finite 2-torsion is our Lemma 3.5. □

Lemma 4.3. *Assuming the conjecture* $\mathbf{L}^c(X_{\acute{e}t}, n)$*, we have the following cohomology:*

groups	***type***	$i \ll 0$	$i \gg 0$
$H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$	*finitely generated*	0	*finite* 2*-torsion*
$\widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n))$	*cofinite type*	*finite* 2*-torsion*	*0*
$H^i_c(X_{\acute{e}t}, \mathbb{Z}(n))$	*cofinite type*	0	*finite* 2*-torsion*

In particular, $R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n))$ *is an almost perfect complex, while* $R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n))$ *is almost of cofinite type in the sense of Definition 1.1.*

Proof. The groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are finitely generated by the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, so the duality Theorem I implies that $\widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n))$ are of cofinite type. The same holds for $H^i_c(X_{\acute{e}t}, \mathbb{Z}(n))$, since they differ by finite 2-torsion according to Lemma 4.2.

By Lemma 4.1 we have $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) = 0$ for $i \ll 0$, and therefore $\widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) = 0$ for $i \gg 0$ by duality, and then $H^i_c(X_{\acute{e}t}, \mathbb{Z}(n))$ is finite 2-torsion for $i \gg 0$ (again using Lemma 4.2).

For $i \ll 0$ we have $H^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) = 0$, so that $\widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n))$ is finite 2-torsion. By duality, $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ is finite 2-torsion for $i \gg 0$. □

5 Complexes $R\Gamma_{fg}(X, \mathbb{Z}(n))$

Definition 5.1. Assuming the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, consider a morphism $\alpha_{X,n}$ in the derived category $\mathbf{D}(\mathbb{Z})$ given by the composition

$$\begin{array}{ccc}
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & \xrightarrow{\mathbb{Q}\twoheadrightarrow\mathbb{Q}/\mathbb{Z}} & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]) \\
 & & \uparrow \text{Theorem I}\ \cong \\
 & \searrow^{\alpha_{X,n}} & R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n)) \\
 & & \downarrow \text{proj.} \\
 & & R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n))
\end{array}$$

Here the first arrow is induced by the canonical projection $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$, and the last arrow is the canonical projection from the modified cohomology with compact support to the usual cohomology with compact support (see Appendix B).

We define the complex $R\Gamma_{fg}(X, \mathbb{Z}(n))$ as a cone of $\alpha_{X,n}$:

$$R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \xrightarrow{\alpha_{X,n}} R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \to R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1])$$

Remark 5.2. We note that our $R\Gamma_{fg}(X, \mathbb{Z}(n))$ plays the same role as $R\Gamma_W(\overline{X}_{\acute{e}t}, \mathbb{Z}(n))$ that appears in [FM2018, Definition 3.6]. We use a different notation, since Flach and Morin work with Artin–Verdier topology, and their complex $R\Gamma_W(\overline{X}_{\acute{e}t}, \mathbb{Z}(n))$ is perfect, while for our complex $H^i_{fg}(X, \mathbb{Z}(n))$ may be finite 2-torsion in arbitrarily high degree.

We first note that although the definition might seem complicated at first, it simplifies if X has no real places.

Proposition 5.3. *If $X(\mathbb{R}) = \varnothing$, then*

$$R\Gamma_{fg}(X, \mathbb{Z}(n)) \cong R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z}[-1]).$$

Proof. In this case $R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n))$ is the identity morphism, and therefore $\alpha_{X,n}$ sits in the following commutative diagram with distinguished columns:

$$\begin{array}{ccc}
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & \xrightarrow{\mathrm{id}} & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \\
\downarrow \alpha_{X,n} & & \downarrow \\
R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) & \xrightarrow[\text{Theorem I}]{\cong} & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]) \\
\downarrow & & \downarrow \\
R\Gamma_{fg}(X, \mathbb{Z}(n)) & \overset{\cong}{\dashrightarrow} & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z}[-1]) \\
\downarrow & & \downarrow \\
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) & \xrightarrow{\mathrm{id}} & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1])
\end{array}$$

Here the first column is our definition of $R\Gamma_{fg}(X, \mathbb{Z}(n))$, and the second column is induced by the distinguished triangle $\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to \mathbb{Z}[1]$. ☐

Proposition 5.4. *Assuming the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, the complex $R\Gamma_{fg}(X, \mathbb{Z}(n))$ is almost perfect in the sense of 1.1, i.e. its cohomology groups $H^i_{fg}(X, \mathbb{Z}(n)) := H^i(R\Gamma_{fg}(X, \mathbb{Z}(n)))$ are finitely generated, trivial for $i \ll 0$, and only have 2-torsion for $i \gg 0$.*

Proof. By the definition of $R\Gamma_{fg}(X, \mathbb{Z}(n))$, we have a long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) \xrightarrow{H^i(\alpha_{X,n})} H^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \longrightarrow H^i_{fg}(X, \mathbb{Z}(n)) \xrightarrow{\delta^i} \mathrm{Hom}(H^{1-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) \xrightarrow{H^{i+1}(\alpha_{X,n})} H^{i+1}_c(X_{\acute{e}t}, \mathbb{Z}(n)) \longrightarrow \cdots$$

First we observe what happens for $|i| \gg 0$. For $i \ll 0$ we have $H^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) = 0$, and therefore

$$H^i_{fg}(X, \mathbb{Z}(n)) \cong \mathrm{Hom}(H^{1-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) = 0,$$

since the group $H^{1-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))$ is torsion by Lemma 4.3. Similarly, the complex $R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n))$ is bounded from below by Lemma 4.1, and therefore for $i \gg 0$ we have $\mathrm{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) = 0$, so that $H^i_{fg}(X, \mathbb{Z}(n)) \cong H^i_c(X_{\acute{e}t}, \mathbb{Z}(n))$, which is finite 2-torsion by Lemma 4.3.

Now we consider short exact sequences

$$0 \longrightarrow \ker \delta^i \longrightarrow H^i_{fg}(X, \mathbb{Z}(n)) \longrightarrow \operatorname{im} \delta^i \longrightarrow 0$$

with $\ker \delta^i = \operatorname{coker} H^i(\alpha_{X,n})$ and $\operatorname{im} \delta^i = \ker H^{i+1}(\alpha_{X,n})$.

By the definition of $\alpha_{X,n}$, the morphism $H^i(\alpha_{X,n})$ factors as

$$\mathrm{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) \to \mathrm{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to H^i_c(X_{\acute{e}t}, \mathbb{Z}(n))$$

We recall from Lemma 4.2 that the morphism $\widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to H^i_c(X_{\acute{e}t}, \mathbb{Z}(n))$ has finite 2-torsion kernel and cokernel.

The group $H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))$ is finitely generated according to the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$. If this group is of the form $\mathbb{Z}^{\oplus r} \oplus T$, the morphism $H^i(\alpha_{X,n})$ is given by

$$\mathbb{Q}^{\oplus r} \twoheadrightarrow (\mathbb{Q}/\mathbb{Z})^{\oplus r} \rightarrowtail \widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to H^i_c(X_{\acute{e}t}, \mathbb{Z}(n))$$

where $(\mathbb{Q}/\mathbb{Z})^{\oplus r} \rightarrowtail \widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n))$ is the inclusion of the maximal divisible subgroup in the group of cofinite type

$$\widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \cong \mathrm{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}).$$

Both kernel and cokernel of the above map are finitely generated, hence $H^i_{fg}(X, \mathbb{Z}(n))$ is finitely generated. ☐

Proposition 5.5. *The complex $R\Gamma_{fg}(X, \mathbb{Z}(n))$ is defined up to a unique isomorphism in the derived category $\mathbf{D}(\mathbb{Z})$.*

Proof. The complex $R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2])$ consists of $\mathbb{Q}$-vector spaces, and $R\Gamma_{fg}(X, \mathbb{Z}(n))$ is almost perfect, so we are in the situation of A.3. ☐

Proposition 5.6. *Assume the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$ holds and consider the distinguished triangle defining $R\Gamma_{fg}(X, \mathbb{Z}(n))$:*

$$R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \xrightarrow{\alpha_{X,n}} R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \xrightarrow{f} R\Gamma_{fg}(X, \mathbb{Z}(n)) \xrightarrow{g} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1])$$

1) The morphism g induces an isomorphism

$$g \otimes \mathbb{Q} \colon R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]).$$

2) For each $m = 1, 2, 3$ the morphism f induces an isomorphism

$$f \otimes \mathbb{Z}/m\mathbb{Z} \colon R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \otimes^{\mathbf{L}}_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \xrightarrow{\cong} R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes^{\mathbf{L}}_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$$

3) For any prime ℓ the morphism f induces an isomorphism

$$\varprojlim_r H_c(X_{\acute{e}t}, \mathbb{Z}/\ell^r(n)) \cong H_{fg}(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_\ell.$$

Proof. The cohomology groups $H^i_c(X_{\acute{e}t}, \mathbb{Z}(n))$ are all torsion, and therefore one has $R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong 0$ in the derived category. Similarly, the complexes $R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[\cdots])$ consist of $\mathbb{Q}$-vector spaces, so they are killed by tensoring with $\mathbb{Z}/m\mathbb{Z}$. This proves 1) and 2).

Now 2) implies 3): by finite generation of $H^i_{fg}(X, \mathbb{Z}(n))$, we have

$$\varprojlim_r H_c(X_{\acute{e}t}, \mathbb{Z}/\ell^r(n)) \overset{2)}{\cong} \varprojlim_r H_{fg}(X, \mathbb{Z}/\ell^r(n)) \cong \varprojlim_r H_{fg}(X, \mathbb{Z}(n))/\ell^r \cong H^i_{fg}(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_\ell. \qquad \square$$

6 Proof of Theorem II

The goal of this section is to prove Theorem II. We recall that it states that the morphism of complexes u^*_∞, defined as the composition

$$\begin{array}{ccc}
R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) & \overset{u^*_\infty}{\dashrightarrow} & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \\
\| & & \uparrow \\
R\Gamma_c(X_{\acute{e}t}, \mathbb{Q}/\mathbb{Z}(n))[-1] & \xrightarrow{v^*_\infty[-1]} & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))[-1]
\end{array}$$

is torsion. Here the morphism $v^*_\infty \colon R\Gamma_c(X_{\acute{e}t}, \mathbb{Q}/\mathbb{Z}(n)) \to R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))$ is induced by the comparison functor $\alpha^* \colon \mathbf{Sh}(X_{\acute{e}t}) \to \mathbf{Sh}(G_{\mathbb{R}}, X(\mathbb{C}))$, as explained in Proposition B.3. We first make sure that α^* identifies the sheaf $\mathbb{Q}/\mathbb{Z}(n)$ on $X_{\acute{e}t}$ from Definition 1.2 with the $G_{\mathbb{R}}$-equivariant sheaf $\mathbb{Q}/\mathbb{Z}(n) := \frac{(2\pi i)^n\,\mathbb{Q}}{(2\pi i)^n\,\mathbb{Z}}$ on $X(\mathbb{C})$.

Proposition 6.1. *Consider the comparison morphism $\alpha^* \colon \mathbf{Sh}(X_{\acute{e}t}) \to \mathbf{Sh}(G_{\mathbb{R}}, X(\mathbb{C}))$, as described in Appendix B. For the sheaf $\mathbb{Q}/\mathbb{Z}(n)$ on $X_{\acute{e}t}$ (see Definition 1.2) we have an isomorphism of $G_{\mathbb{R}}$-equivariant constant sheaves on $X(\mathbb{C})$*

$$\alpha^* \mathbb{Q}/\mathbb{Z}(n) \cong \mathbb{Q}/\mathbb{Z}(n).$$

Proof. First of all, since α^* is the composition of certain inverse image functors γ^* and ϵ^* (which are left adjoint) and an equivalence of categories δ_*, the functor α^* preserves colimits, and in particular

$$\alpha^* \mathbb{Q}/\mathbb{Z}(n) \cong \bigoplus_p \varinjlim_r \alpha^* j_{p!} \mu^{\otimes n}_{p^r}. \tag{6.1}$$

Another formal observation is that the base change from $\operatorname{Spec} \mathbb{Z}$ to $\operatorname{Spec} \mathbb{C}$ factors through the base change to $\operatorname{Spec} \mathbb{Z}[1/p]$, and then $j^*_p \circ j_{p!} = id_{\mathbf{Sh}(X[1/p]_{\acute{e}t})}$:

$$\begin{array}{ccccc}
\mathbf{Sh}(X[1/p]_{\acute{e}t}) & \xrightarrow{j_{p!}} & \mathbf{Sh}(X_{\acute{e}t}) & \xrightarrow{\gamma^*} & \mathbf{Sh}(X_{\mathbb{C},\acute{e}t}) \\
& \searrow^{id} & \downarrow^{j^*_p} & \nearrow & \\
& & \mathbf{Sh}(X[1/p]_{\acute{e}t}) & &
\end{array}$$

which means that we may safely erase "$j_{p!}$" in (6.1), and everything boils down to calculating the sheaves

$$\alpha^* \mu_{p^r}^{\otimes n} = \alpha^* \underline{\mathrm{Hom}}_{X[1/p]}(\mu_{p^r}^{\otimes(-n)}, \mathbb{Z}/p^r\mathbb{Z}).$$

As we base change to $\operatorname{Spec}\mathbb{C}$, the étale sheaf μ_{p^r} simply becomes the constant sheaf $\mu_{p^r}(\mathbb{C})$ on $X(\mathbb{C})$, and

$$\alpha^* \mu_{p^r}^{\otimes n} = \underline{\mathrm{Hom}}_{X(\mathbb{C})}(\mu_{p^r}^{\otimes(-n)}(\mathbb{C}), \mathbb{Z}/p^r\mathbb{Z}).$$

Here the twist is given by

$$\mu_m(\mathbb{C})^{\otimes(-n)} := \underbrace{\mu_m(\mathbb{C}) \otimes \cdots \otimes \mu_m(\mathbb{C})}_{-n},$$

with the $G_\mathbb{R}$-action on tensor products $A \otimes B$ defined as usual by $g \cdot (a \otimes b) = g \cdot a \otimes g \cdot b$, and the $G_\mathbb{R}$-action on $\underline{\mathrm{Hom}}(A, B)$ being $(g \cdot f)(a) := g \cdot f(g^{-1} \cdot a)$.

What follows are well-known calculations, and we just need to take care of the actions of $G_\mathbb{R}$ and make sure that everything is equivariant. First we see that there is a canonical isomorphism of $G_\mathbb{R}$-modules

$$\mu_m(\mathbb{C}) \cong \frac{(2\pi i)\,\mathbb{Z}}{m\,(2\pi i)\,\mathbb{Z}}, \quad e^{2\pi i k/m} \mapsto 2\pi i k. \tag{6.2}$$

Now there is a $G_\mathbb{R}$-isomorphism

$$\underbrace{(2\pi i)\,\mathbb{Z} \otimes \cdots \otimes (2\pi i)\,\mathbb{Z}}_{-n} \xrightarrow{\cong} (2\pi i)^{-n}\,\mathbb{Z}, \tag{6.3}$$
$$(2\pi i)\,a_1 \otimes \cdots \otimes (2\pi i)\,a_{-n} \mapsto (2\pi i)^{-n}\,a_1 \cdots a_{-n},$$

and combining (6.2) and (6.3), we obtain

$$\mu_m(\mathbb{C})^{(-n)} \cong \frac{(2\pi i)^{-n}\,\mathbb{Z}}{m\,(2\pi i)^{-n}\,\mathbb{Z}}.$$

Finally, we have $G_\mathbb{R}$-isomorphisms

$$\underline{\mathrm{Hom}}(\mu_m(\mathbb{C})^{(-n)}, \mathbb{Z}/m\mathbb{Z}) \cong \underline{\mathrm{Hom}}\Big(\frac{(2\pi i)^{-n}\,\mathbb{Z}}{m\,(2\pi i)^{-n}\,\mathbb{Z}}, \mathbb{Z}/m\mathbb{Z}\Big) \cong \underline{\mathrm{Hom}}((2\pi i)^{-n}\,\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \frac{(2\pi i)^n\,\mathbb{Z}}{m\,(2\pi i)^n\,\mathbb{Z}},$$

where the last isomorphism is given by $f \mapsto (2\pi i)^n\, f((2\pi i)^{-n} \cdot 1)$. Now

$$\alpha^*\mathbb{Z}(n) \cong \bigoplus_p \varinjlim_r \mu_{p^r}(\mathbb{C})^{\otimes n} \cong \bigoplus_p \varinjlim_r \frac{(2\pi i)^n\,\mathbb{Z}}{p^r\,(2\pi i)^n\,\mathbb{Z}} \cong \frac{(2\pi i)^n\,\mathbb{Q}}{(2\pi i)^n\,\mathbb{Z}}.$$

This is a colimit of $G_\mathbb{R}$-modules, since the transition morphisms are $G_\mathbb{R}$-equivariant. ☐

We proceed with our proof of Theorem II. This seems to be rather nontrivial; our argument (motivated by [FM2018] where it is given under the assumption that X is proper and regular) will be based on the following result about ℓ-adic cohomology.

Proposition 6.2. *Let $f\colon X \to \operatorname{Spec}\mathbb{Z}$ be an arithmetic scheme (that is, with f separated, of finite type) and $n < 0$. Then for any prime ℓ we have*

$$(H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_\mathbb{Q}})_{div} = 0.$$

Proof. Let us recall some facts about ℓ-adic cohomology. We refer to [SGA 5, Exposé VI] for the details. Let us first consider the sheaf $\mathbb{Z}_\ell(n)$. It is a **constructible $\mathbb{Z}_\ell$-sheaf*** on X in the sense of [SGA 5, Exposé VI, 1.1.1]. We would like to compare the cohomology of $\mathbb{Z}_\ell(n)$ on $X_{\overline{\mathbb{Q}},\acute{e}t}$ and $X_{\overline{\mathbb{F}_p},\acute{e}t}$, where p is some prime different from ℓ, to be determined later. For this we fix some algebraic closures $\overline{\mathbb{Q}}/\mathbb{Q}$ and $\overline{\mathbb{F}_p}/\mathbb{F}_p$ and consider the corresponding morphisms

$$\overline{\eta}\colon \operatorname{Spec}\overline{\mathbb{Q}} \to \operatorname{Spec}\mathbb{Z}, \quad \overline{x}\colon \operatorname{Spec}\overline{\mathbb{F}_p} \to \operatorname{Spec}\mathbb{Z}.$$

Let $X_{\overline{\mathbb{Q}},\acute{e}t}$ and $X_{\overline{\mathbb{F}_p},\acute{e}t}$ be the pullbacks of X along the above morphisms:

$$\begin{array}{ccccc} X_{\overline{\mathbb{Q}}} & \longrightarrow & X & \longleftarrow & X_{\overline{\mathbb{F}_p}} \\ \downarrow{\scriptstyle f_{\overline{\mathbb{Q}}}} & \lrcorner & \downarrow{\scriptstyle f} & \llcorner & \downarrow{\scriptstyle f_{\overline{\mathbb{F}_p}}} \\ \operatorname{Spec}\overline{\mathbb{Q}} & \xrightarrow[\overline{\eta}]{} & \operatorname{Spec}\mathbb{Z} & \xleftarrow[\overline{x}]{} & \operatorname{Spec}\overline{\mathbb{F}_p} \end{array}$$

According to [SGA 5, Exposé VI, 2.2.3], the proper base change theorem holds for constructible $\mathbb{Z}_\ell$-sheaves. It gives us isomorphisms

$$H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t}, \mathbb{Z}_\ell(n)) \cong (R^i f_! \mathbb{Z}_\ell(n))_{\overline{\eta}}, \quad H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t}, \mathbb{Z}_\ell(n)) \cong (R^i f_! \mathbb{Z}_\ell(n))_{\overline{x}},$$

where $R^i f_! \mathbb{Z}_\ell(n)$ is the same sheaf on $\operatorname{Spec}\mathbb{Z}$, and we take its different stalks to get cohomology with compact support on different fibers. The construction of higher direct images with proper support $R^i f_! \mathcal{F}$ for ℓ-adic sheaves is given in [SGA 5, Exposé VI, §2.2]. The key nontrivial fact that we need is that for every morphism (of locally noetherian schemes) $f\colon X \to Y$, separated of finite type, if $\mathcal{F}$ is a constructible $\mathbb{Z}_\ell$-sheaf on X, then $R^i f_! \mathcal{F}$ is a constructible $\mathbb{Z}_\ell$-sheaf on Y.

According to [SGA 5, Exposé VI, 1.2.6], for a projective system of abelian sheaves $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ on $X_{\acute{e}t}$, the following are equivalent:

1) $\mathcal{F}$ is a constructible $\mathbb{Z}_\ell$-sheaf,

2) every open subscheme $U \subset X$ is a finite union of locally closed pieces Z_i where $\mathcal{F}|_{Z_i}$ is a **twisted constant constructible $\mathbb{Z}_\ell$-sheaf****.

Being "twisted constant" means that each sheaf $\mathcal{F}_n$ in the projective system $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is locally constant. The importance of twisted constant sheaves is explained by the following property [SGA 5, Exposé VI, 1.2.4, 1.2.5]: for a connected locally noetherian scheme X, the category of twisted constant $\mathbb{Z}_\ell$-constructible sheaves on X is equivalent to the category of finitely generated $\mathbb{Z}_\ell$-modules with a continuous action of the étale fundamental group $\pi_1^{\acute{e}t}(X)$.

In our setting, all this means that there exists an open subscheme

$$U = \operatorname{Spec}\mathbb{Z}_S \subset \operatorname{Spec}\mathbb{Z},$$

where $\mathbb{Z}_S$ denotes the localization of $\mathbb{Z}$ at a finite set of primes S, such that the sheaves $R^i f_! \mathbb{Z}_\ell(n)$ are twisted constant on U. By removing the necessary bad primes, we can make sure this holds for all i.

Now there exists some prime $p \notin S$ (that is, $(p) \in U$), for which we may consider the following picture:

$$\begin{array}{ccccc} X_{\overline{\mathbb{Q}}} & \longrightarrow & X_U & \longleftarrow & X_{\overline{\mathbb{F}_p}} \\ \downarrow{\scriptstyle f_{\overline{\mathbb{Q}}}} & \lrcorner & \downarrow{\scriptstyle f_U} & \llcorner & \downarrow{\scriptstyle f_{\overline{\mathbb{F}_p}}} \\ \operatorname{Spec}\overline{\mathbb{Q}} & \xrightarrow[\overline{\eta}]{} & U & \xleftarrow[\overline{x}]{} & \operatorname{Spec}\overline{\mathbb{F}_p} \end{array}$$

*Or simply $\mathbb{Z}_\ell$-**sheaf** in the terminology of [SGA $4\frac{1}{2}$, Rapport].

A **faisceau lisse in the terminology of [SGA $4\frac{1}{2}$, Rapport].

It follows that we have isomorphisms

$$H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t},\mathbb{Z}_\ell(n)) \cong (R^if_{U,!}\mathbb{Z}_\ell(n))_{\overline{\eta}} \cong (R^if_{U,!}\mathbb{Z}_\ell(n))_{\overline{x}} \cong H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Z}_\ell(n)), \tag{6.4}$$

of finitely generated $\mathbb{Z}_\ell$-modules with continuous action of

$$\pi_1^{\acute{e}t}(U) \cong \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q}),$$

where $\mathbb{Q}_S/\mathbb{Q}$ denotes a maximal extension of $\mathbb{Q}$ unramified outside of S. We note that $(R^if_{U,!}\mathbb{Z}_\ell(n))_{\overline{\eta}}$ naturally carries an action of $\pi_1^{\acute{e}t}(U,\overline{\eta})$, while $(R^if_{U,!}\mathbb{Z}_\ell(n))_{\overline{x}}$ carries an action of $\pi_1^{\acute{e}t}(U,\overline{x})$, and the isomorphism in the middle of (6.4) sweeps under the rug an identification of $\pi_1^{\acute{e}t}(U,\overline{\eta})$ with $\pi_1^{\acute{e}t}(U,\overline{x})$.

To state this more accurately, note that the $\mathbb{Z}_\ell$-module $H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t},\mathbb{Z}_\ell(n))$ carries a natural action of $G_{\mathbb{Q}}$, while $H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Z}_\ell(n))$ carries a natural action of $G_{\mathbb{F}_p}$. After making the necessary choices, we have $G_{\mathbb{Q}_p} \subset G_{\mathbb{Q}}$ and a short exact sequence

$$1 \to I_p \to G_{\mathbb{Q}_p} \to G_{\mathbb{F}_p} \to 1$$

where I_p is the inertia subgroup, acting trivially on $H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t},\mathbb{Z}_\ell(n))$. We have thus isomorphisms of finitely generated $\mathbb{Z}_\ell$-modules

$$H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t},\mathbb{Z}_\ell(n)) \cong H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Z}_\ell(n)),$$

equivariant under the action of $G_{\mathbb{Q}_p}/I_p$ on the left hand side and of $G_{\mathbb{F}_p}$ on the right hand side. To relate all this to $\mathbb{Q}_\ell(n)$ and $\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)$-coefficients, note that we have the following isomorphic long exact sequences in cohomology.

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
H^{i-1}_c(X_{\overline{\mathbb{Q}},\acute{e}t},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) & \xrightarrow{\cong} & H^{i-1}_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\
H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t},\mathbb{Z}_\ell(n)) & \xrightarrow{\cong} & H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Z}_\ell(n)) \\
\downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\
H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t},\mathbb{Q}_\ell(n)) & \xrightarrow{\cong} & H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Q}_\ell(n)) \\
\downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} \\
H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) & \longrightarrow & H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \\
\downarrow{\scriptstyle\cong} & & \downarrow \\
\vdots & & \vdots
\end{array} \tag{6.5}$$

Here

$$H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t},\mathbb{Q}_\ell(n)) = H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t},\mathbb{Z}_\ell(n)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,$$
$$H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Q}_\ell(n)) = H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Z}_\ell(n)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,$$

and the arrows ϕ above are canonical localization morphisms. The horizontal arrows are equivariant isomorphisms in the above sense. Note that we have

$$H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{Q}}} \rightarrowtail H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{Q}_p}/I_p} \cong H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{F}_p}},$$

so in order to prove that

$$(H^i_c(X_{\overline{\mathbb{Q}},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_\mathbb{Q}})_{div} = 0,$$

it will be enough to show that

$$(H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{F}_p}})_{div} = 0.$$

From now on we move to the characteristic p and consider the fixed points of $G_{\mathbb{F}_p}$ acting on the $\mathbb{Z}_\ell$-module $H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$. In the long exact sequence (6.5), we have (keeping in mind that ϕ is merely the localization morphism):

$$\begin{aligned} \ker\phi &= H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Z}_\ell(n))_{tor}, \\ \ker\psi &= \operatorname{im}\phi \cong H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Z}_\ell(n))/\ker\phi \\ &= \frac{H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Z}_\ell(n))}{H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Z}_\ell(n))_{tor}} =: H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Z}_\ell(n))_{cotor}, \\ \operatorname{im}\psi &= H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div}. \end{aligned}$$

This gives us a short exact sequence

$$0 \to H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Z}_\ell(n))_{cotor} \to H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell(n)) \to H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div} \to 0$$

After taking the $G_{\mathbb{F}_p}$-invariants, we obtain a long exact sequence of cohomology groups

$$\begin{aligned} 0 \to (H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Z}_\ell(n))_{cotor})^{G_{\mathbb{F}_p}} &\to H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell(n))^{G_{\mathbb{F}_p}} \\ &\to (H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div})^{G_{\mathbb{F}_p}} \to H^1(G_{\mathbb{F}_p}, H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Z}_\ell(n))_{cotor}) \to \cdots \end{aligned} \quad (6.6)$$

We claim that

$$H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell(n))^{G_{\mathbb{F}_p}} = 0. \quad (6.7)$$

Indeed, according to [SGA 7, Exposé XXI, 5.5.3], the eigenvalues of the geometric Frobenius acting on $H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell)$ are algebraic integers. We are twisting $\mathbb{Q}_\ell$ by n, so the eigenvalues of Frobenius lie in $p^{-n}\,\overline{\mathbb{Z}}$. Since $n < 0$ by our assumption, this implies that 1 does not occur as an eigenvalue.

Now (6.7) and the long exact sequence (6.6) imply that there is a monomorphism

$$(H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div})^{G_{\mathbb{F}_p}} \rightarrowtail H^1(G_{\mathbb{F}_p}, H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Z}_\ell(n))_{cotor}),$$

which restricts to a monomorphism between the maximal divisible subgroups

$$((H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div})^{G_{\mathbb{F}_p}})_{div} \rightarrowtail H^1(G_{\mathbb{F}_p}, H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Z}_\ell(n))_{cotor})_{div}.$$

However, $H^1(G_{\mathbb{F}_p}, H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Z}_\ell(n))_{cotor})$ is a finitely generated $\mathbb{Z}_\ell$-module, and therefore its maximal divisible subgroup is trivial. We have therefore

$$(H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{F}_p}})_{div} = ((H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div})^{G_{\mathbb{F}_p}})_{div} = 0.$$

(For the first equality, note that for any G-module A one has $((A_{div})^G)_{div} = (A^G)_{div}$.) ☐

Proof of Theorem II. By Definition 1.3, this amounts to showing that the morphism

$$v^*_\infty \colon R\Gamma_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))$$

is torsion. The complexes $R\Gamma_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n))$ and $R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))$ are almost of cofinite type by Lemma 4.3 and 3.7 respectively. Therefore, according to A.4, to show that $v^*_\infty \colon R\Gamma_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n)) \to$

$R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))$ is torsion in $\mathbf{D}(\mathbb{Z})$, it is enough to show that the corresponding morphisms on the maximal divisible subgroups

$$H^i_c(v^*_\infty)_{div} \colon H^i_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n))_{div} \to H^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))_{div}$$

are all trivial. The morphism $H^i_c(v^*_\infty)$ factors through $H^i_c(X_{\overline{\mathbb{Q}},ét}, \mu^{\otimes n})^{G_{\mathbb{Q}}}$, where $\mu^{\otimes n}$ is the sheaf of all roots of unity on $X_{\overline{\mathbb{Q}},ét}$ twisted by n. We have therefore

$$\begin{array}{ccc} H^i_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n))_{div} & \xrightarrow{H^i_c(v^*_\infty)_{div}} & H^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))_{div} \\ & \searrow \quad \nearrow & \\ & \left(H^i_c(X_{\overline{\mathbb{Q}},ét}, \mu^{\otimes n})^{G_{\mathbb{Q}}}\right)_{div} & \end{array}$$

Now

$$\left(H^i_c(X_{\overline{\mathbb{Q}},ét}, \mu^{\otimes n})^{G_{\mathbb{Q}}}\right)_{div} \cong \left(\bigoplus_\ell H^i_c(X_{\overline{\mathbb{Q}},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{Q}}}\right)_{div} \cong \bigoplus_\ell \left(H^i_c(X_{\overline{\mathbb{Q}},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{Q}}}\right)_{div},$$

where all summands are trivial according to 6.2. ☐

7 Weil-étale complexes $R\Gamma_{W,c}(X, \mathbb{Z}(n))$

The goal of this section is to construct Weil-étale cohomology complexes $R\Gamma_{W,c}(X, \mathbb{Z}(n))$.

Lemma 7.1. *Let X be an arithmetic scheme and $n < 0$. Assume the conjecture $\mathbf{L}^c(X_{ét}, n)$, so that the morphism $\alpha_{X,n}$ exists. Then $u^*_\infty \circ \alpha_{X,n} = 0$.*

$$\begin{array}{ccc} R\mathrm{Hom}(R\Gamma(X, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & & \\ \downarrow{\scriptstyle \alpha_{X,n}} & \overset{=0}{\searrow} & \\ R\Gamma_c(X_{ét}, \mathbb{Z}(n)) & \xrightarrow{u^*_\infty} & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \end{array}$$

Proof. The morphism $\alpha_{X,n}$ is defined on the complex of $\mathbb{Q}$-vector spaces $R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-2])$, and u^*_∞ is torsion by Theorem II. ☐

Definition 7.2. We let $i^*_\infty \colon R\Gamma_{fg}(X, \mathbb{Z}(n)) \to R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))$ be a morphism in $\mathbf{D}(\mathbb{Z})$ that gives a morphism of distinguished triangles

$$\begin{array}{ccc} R\mathrm{Hom}(R\Gamma(X, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \alpha_{X,n}} & & \downarrow \\ R\Gamma_c(X_{ét}, \mathbb{Z}(n)) & \xrightarrow{u^*_\infty} & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \\ \downarrow & & \downarrow{\scriptstyle id} \\ R\Gamma_{fg}(X, \mathbb{Z}(n)) & \overset{i^*_\infty}{\dashrightarrow} & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \\ \downarrow & & \downarrow \\ R\mathrm{Hom}(R\Gamma(X, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) & \longrightarrow & 0 \end{array} \tag{7.1}$$

Proposition 7.3. *The morphism i^*_∞ is uniquely defined.*

Proof. We may apply A.3, since $R\mathrm{Hom}(R\Gamma(X, \mathbb{Z}^c(n)), \mathbb{Q}[-2])$ is a complex of $\mathbb{Q}$-vector spaces, and both $R\Gamma_{fg}(X, \mathbb{Z}(n))$ and $R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))$ are almost perfect complexes by Proposition 5.4 and Lemma 3.6. □

Proposition 7.4. *The morphism i_∞^* is torsion in the derived category, i.e. $i_\infty^* \otimes \mathbb{Q} = 0$.*

Proof. Let us examine the morphism of distinguished triangles (7.1) that defines i_∞^*; in particular, the commutative diagram

$$\begin{array}{ccc} R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X, \mathbb{Z}(n)) \\ {\scriptstyle u_\infty^*}\downarrow & \swarrow {\scriptstyle i_\infty^*} & \\ R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) & & \end{array}$$

According to A.3, the morphism

$$\mathrm{Hom}_{\mathbf{D}(\mathbb{Z})}(R\Gamma_{fg}(X, \mathbb{Z}(n)), R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))) \to \mathrm{Hom}_{\mathbf{D}(\mathbb{Z})}(R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)), R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)))$$

induced by the composition with $R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to R\Gamma_{fg}(X, \mathbb{Z}(n))$, is mono, and therefore

$$\mathrm{Hom}_{\mathbf{D}(\mathbb{Z})}(R\Gamma_{fg}(X, \mathbb{Z}(n)), R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))) \otimes_\mathbb{Z} \mathbb{Q} \to \\ \mathrm{Hom}_{\mathbf{D}(\mathbb{Z})}(R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)), R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))) \otimes_\mathbb{Z} \mathbb{Q}$$

is mono as well. However, $u_\infty^* \otimes \mathbb{Q} = 0$ by Theorem II, and this implies that $i_\infty^* \otimes \mathbb{Q} = 0$. □

Now we are ready to define Weil-étale complexes.

Definition 7.5. We let $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ be an object in the derived category $\mathbf{D}(\mathbb{Z})$ which is a mapping fiber of i_∞^*:

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \xrightarrow{i_\infty^*} R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \to R\Gamma_{W,c}(X, \mathbb{Z}(n))[1]$$

The **Weil-étale cohomology with compact support** is given by

$$H^i_{W,c}(X, \mathbb{Z}(n)) := H^i(R\Gamma_{W,c}(X, \mathbb{Z}(n))).$$

Remark 7.6. Note that this defines $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ up to a non-unique isomorphism in $\mathbf{D}(\mathbb{Z})$, and the groups $H^i_{W,c}(X, \mathbb{Z}(n))$ are also defined up to a non-unique isomorphism. In a continuation of this paper we will make use of the determinant $\det_\mathbb{Z} R\Gamma_{W,c}(X, \mathbb{Z}(n))$ (in the sense of [KM1976]), which will be defined up to a canonical isomorphism.

Nevertheless, we recall from Proposition 5.5 that $R\Gamma_{fg}(X, \mathbb{Z}(n))$ is defined up to a unique isomorphism in the derived category $\mathbf{D}(\mathbb{Z})$. If we could define $i_\infty^* \colon R\Gamma_{fg}(X, \mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))$ as an explicit, genuine morphism of complexes (not merely a morphism in the derived category $\mathbf{D}(\mathbb{Z})$), this would give us a canonical and functorial definition for $R\Gamma_{W,c}(X, \mathbb{Z}(n))$.

Proposition 7.7. *The conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$ implies that $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ is a perfect complex.*

Proof. By definition, we have a long exact sequence in cohomology

$$\cdots \to H^{i-1}_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \to H^i_{W,c}(X, \mathbb{Z}(n)) \to H^i_{fg}(X, \mathbb{Z}(n)) \xrightarrow{H^i(i_\infty^*)} H^i_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \to \cdots$$

The groups $H^i_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))$ and $H^i_{fg}(X, \mathbb{Z}(n))$ are finitely generated by Lemma 3.6 and Proposition 5.4. They vanish for $i \ll 0$, but they are finite 2-torsion for $i \gg 0$. I claim that $H^i(i_\infty^*)$ is an isomorphism for $i \gg 0$, meaning that this 2-torsion in higher degrees does not appear in $H^i_{W,c}(X, \mathbb{Z}(n))$. We have a commutative diagram

$$\begin{array}{ccc} H^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) & \longrightarrow & H^i_{fg}(X, \mathbb{Z}(n)) \\ {\scriptstyle H^i(u_\infty^*)}\downarrow & \swarrow {\scriptstyle H^i(i_\infty^*)} & \\ H^i_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) & & \end{array}$$

The morphism $H^i(u_\infty^*)$ is iso for $i \gg 0$, hence $H^i(i_\infty^*)$ is surjective for $i \gg 0$. However, $H^i_{fg}(X, \mathbb{Z}(n))$ and $H^i_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))$ have the same 2-torsion for $i \gg 0$, and $H^i(i_\infty^*)$ is iso for $i \gg 0$. □

Proposition 7.8. *There is a non-canonical splitting*

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes_\mathbb{Z} \mathbb{Q} \cong R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q})[-1] \oplus R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Q}(n))[-1].$$

Proof. The distinguished triangle defining $R\Gamma_{W,c}(X, \mathbb{Z}(n))$

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \xrightarrow{i_\infty^*} R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \to R\Gamma_{W,c}(X, \mathbb{Z}(n))[1]$$

after tensoring with $\mathbb{Q}$ becomes

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes_\mathbb{Z} \mathbb{Q} \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes_\mathbb{Z} \mathbb{Q} \xrightarrow{i_\infty^* \otimes \mathbb{Q} = 0} R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \otimes_\mathbb{Z} \mathbb{Q} \to R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes_\mathbb{Z} \mathbb{Q}[1]$$

which gives us a non-canonical splitting [Ver1996, Chapitre II, Corollaire 1.2.6]

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes_\mathbb{Z} \mathbb{Q} \cong R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes_\mathbb{Z} \mathbb{Q} \oplus R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))[-1] \otimes_\mathbb{Z} \mathbb{Q},$$

and we already noticed in 5.6 that

$$R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes_\mathbb{Z} \mathbb{Q} \cong R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q})[-1].$$ □

Proposition 7.9. *If the scheme X is of characteristic p (i.e. the morphism $X \to \operatorname{Spec} \mathbb{Z}$ factors through $\operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$), then*

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \cong R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z}[-1]).$$

Proof. Under our assumptions, $X(\mathbb{C}) = X(\mathbb{R}) = \varnothing$, and therefore $R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) = 0$, so that $R\Gamma_{W,c}(X, \mathbb{Z}(n)) \cong R\Gamma_{fg}(X, \mathbb{Z}(n))$. Finally, according to 5.3, we have an isomorphism $R\Gamma_{fg}(X, \mathbb{Z}(n)) \cong R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z}[-1])$. □

8 Known cases of the conjectutre $\mathbf{L}^c(X_{\acute{e}t}, n)$

Since the main constructions of this paper assume the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, here we list some particular cases when it is known to hold, and therefore gives unconditional results. This section follows [Mor2014, §5] very closely. For an arithmetic scheme X, we formulate the following conjecture that is the conjunction of $\mathbf{L}^c(X_{\acute{e}t}, n)$ for all $n < 0$.

Conjecture. $\mathbf{L}^c(X_{\acute{e}t})$: the cohomology groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are finitely generated for all $i \in \mathbb{Z}$ and $n < 0$.

We note that this is similar to the conjecture [Mor2014, Definition 5.8], the only difference being that Morin also requires finite generation of $H^i(X_{\acute{e}t}, \mathbb{Z}^c(0))$ for $i \le 0$.

The conjecture $\mathbf{L}^c(X_{\acute{e}t})$ is known for number rings, and also for certain varieties over finite fields. As in [Sou1984], [Gei2004], and [Mor2014], we consider the following class.

Definition 8.1. Let $A(\mathbb{F}_q)$ be the full subcategory of the category of smooth projective varieties over a finite field $\mathbb{F}_q$ generated by products of curves and the following operations.

1) If X and Y lie in $A(\mathbb{F}_q)$, then $X \sqcup Y$ lies $A(\mathbb{F}_q)$.

2) If Y lies in $A(\mathbb{F}_q)$ and there are morphisms $c\colon X \to Y$ and $c'\colon Y \to X$ in the category of Chow motives such that $c' \circ c\colon X \to X$ is a multiplication by constant, then X lies in $A(\mathbb{F}_q)$.

3) If $\mathbb{F}_{q^m}/\mathbb{F}_q$ is a finite extension and $X_{\mathbb{F}_q^m} = X \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} \mathbb{F}_{q^m}$ lies in $A(\mathbb{F}_{q^m})$, then X lies in $A(\mathbb{F}_q)$.

4) If X and Y lie in $A(\mathbb{F}_q)$, and Y is a closed subscheme of X, then the blowup of X along Y lies in $A(\mathbb{F}_q)$.

Lemma 8.2. *The conjecture $\mathbf{L}^c(X_{ét})$ holds in the following cases.*

1) $X = \operatorname{Spec} \mathcal{O}_F$ for a number field F.

2) X is a variety over a finite field that belongs to $A(\mathbb{F}_q)$.

Proof. For part 1), by [Mor2014, Theorem 5.1 (b)], the groups $H^i(X_{ét}, \mathbb{Z}(n))$ are finitely generated for all $i \in \mathbb{Z}$ and $n \geq 2$, where $\mathbb{Z}(n)$ denotes Bloch's cycle complex. If $X \to \operatorname{Spec} \mathbb{Z}$ is proper of pure dimension d, then $\mathbb{Z}(n) = \mathbb{Z}^c(d-n)[-2d]$. In this particular case $d = 1$ and $n < 0$.

Part 2) is proved in [Mor2014, Proposition 5.7]. □

Lemma 8.3. *As always, let $n < 0$ be a strictly negative integer.*

1) Let X be an arithmetic scheme, $Z \subset X$ a closed subscheme and $U := X \setminus Z$ its open complement. If the conjecture $\mathbf{L}^c(Y_{ét}, n)$ holds for two schemes of $Y = X, Z, U$, then it holds for the third.

2) For a finite disjoint union of arithmetic schemes $X = \coprod_{1 \leq j \leq p} X_j$, the conjecture $\mathbf{L}^c(X_{ét}, n)$ is equivalent to the conjunction of $\mathbf{L}^c(X_{j,ét}, n)$ for all $1 \leq j \leq p$.

3) For the relative affine space $\mathbb{A}^r_X = \mathbb{A}^r \times_{\operatorname{Spec} \mathbb{Z}} X$ the conjectures $\mathbf{L}^c(\mathbb{A}^r_{X,ét}, n)$ and $\mathbf{L}^c(X_{ét}, n-r)$ are equivalent.

4) Let $\{U_i \to X\}_{i \in I}$ be a finite surjective family of étale morphisms. For $(i_0, \ldots, i_p) \in I^{p+1}$ denote $U_{i_0,\ldots,i_p} := U_{i_0} \times_X \cdots \times_X U_{i_p}$. If the conjecture $\mathbf{L}^c(U_{i_0,\ldots,i_p,ét}, n)$ holds for any $(i_0, \ldots, i_p)$, then $\mathbf{L}^c(X_{ét}, n)$ holds as well.

Proof. For part 1), according to [Gei2010, Corollary 7.2], there is a distinguished triangle

$$R\Gamma(Z_{ét}, \mathbb{Z}^c(n)) \to R\Gamma(X_{ét}, \mathbb{Z}^c(n)) \to R\Gamma(U_{ét}, \mathbb{Z}^c(n)) \to R\Gamma(Z_{ét}, \mathbb{Z}^c(n))[1]$$

and the claim follows from the corresponding long exact sequence

$$\cdots \to H^i(Z_{ét}, \mathbb{Z}^c(n)) \to H^i(X_{ét}, \mathbb{Z}^c(n)) \to H^i(U_{ét}, \mathbb{Z}^c(n)) \to H^{i+1}(Z_{ét}, \mathbb{Z}^c(n)) \to \cdots$$

Part 2) is immediate from $H^i(X_{ét}, \mathbb{Z}^c(n)) \cong \bigoplus_{1 \leq j \leq p} H^i(X_{j,ét}, \mathbb{Z}^c(n))$.

Part 3) is a consequence of the isomorphism $H^i(\mathbb{A}^r_{X,ét}, \mathbb{Z}^c(n)) \cong H^{i+2r}(X_{ét}, \mathbb{Z}^c(n-r))$ proved in [Mor2014, Lemma 5.11].

Finally, for part 4), thanks to the Cartan–Leray spectral sequence

$$E_1^{p,q} = \bigoplus_{(i_0,\ldots,i_p) \in I^{p+1}} H^q(U_{i_0,\ldots,i_p,ét}, \mathbb{Z}^c(n)) \Longrightarrow H^{p+q}(X_{ét}, \mathbb{Z}^c(n)),$$

if the groups $H^q(U_{i_0,\ldots,i_p,ét}, \mathbb{Z}^c(n))$ are finitely generated, then $H^i(X_{ét}, \mathbb{Z}^c(n))$ are finitely generated for all $i \in \mathbb{Z}$. (We recall from lemma 4.1 that assuming finite generation, one has $H^q(U_{i_0,\ldots,i_p,ét}, \mathbb{Z}^c(n)) = 0$ for $q \ll 0$.) □

Now following [Mor2014], we consider the following class of schemes.

Definition 8.4. Let $\mathcal{L}(\mathbb{Z})$ be the full subcategory of arithmetic schemes generated by the following objects:

- the empty scheme $\varnothing$,
- $\operatorname{Spec} \mathcal{O}_F$ for a number field F,

- varieties $X \in A(\mathbb{F}_q)$ for any finite field $\mathbb{F}_q$,

and the following operations.

$\mathcal{L}1$) Let X be an arithmetic scheme, $Z \subset X$ a closed subscheme and $U := X \setminus Z$ its open complement. If one of the schemes X, Z, U lies in $\mathcal{L}(\mathbb{Z})$, then the third also lies in $\mathcal{L}(\mathbb{Z})$.

$\mathcal{L}2$) A finite disjoint union $X = \coprod_{1 \le j \le p} X_j$ lies in $\mathcal{L}(\mathbb{Z})$ if and only if each X_j lies in $\mathcal{L}(\mathbb{Z})$.

$\mathcal{L}3$) If $V \to U$ is an affine bundle and U lies in $\mathcal{L}(\mathbb{Z})$, then V also lies in $\mathcal{L}(\mathbb{Z})$.

$\mathcal{L}4$) If $\{U_i \to X\}_{i \in I}$ is a finite surjective family of étale morphisms such that each $U_{i_0,\ldots,i_p}$ lies in $\mathcal{L}(\mathbb{Z})$, then X also lies in $\mathcal{L}(\mathbb{Z})$.

This is similar to the class $\mathcal{L}(\mathbb{Z})$ from [Mor2014, Definition 5.9], with the only difference that Morin requires in $\mathcal{L}1$) that Z is proper and regular.

Proposition 8.5. *The conjecture* $\mathbf{L}^c(X_{\acute{e}t})$ *holds for any arithmetic scheme* $X \in \mathcal{L}(\mathbb{Z})$.

Proof. Follows from lemmas 8.2 and 8.3 (that in fact motivate the definition of $\mathcal{L}(\mathbb{Z})$). ☐

Finally, as in [Mor2014, §5.4], we consider cellular schemes.

Definition 8.6. Let Y be a scheme separated and of finite type over $\operatorname{Spec} k$ for a field k. We say that Y **admits a cellular decomposition** if there exists a filtration of Y by reduced closed subschemes

$$Y^{red} = Y_N \supseteq Y_{N-1} \supseteq \cdots \supseteq Y_{-1} = \varnothing$$

such that $Y_i \setminus Y_{i-1} \cong \mathbb{A}_k^{r_i}$ is isomorphic to an affine space over k.

We say that Y is **geometrically cellular** if $Y_{\overline{k}} = Y \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ admits a cellular decomposition. This is equivalent to existence of a finite Galois extension k'/k such that $Y_{k'}$ admits a cellular decomposition.

Finally, given an S-scheme $X \to S$ that is separated and of finite type, we say that X is **geometrically cellular** if for each $s \in S$ the corresponding fiber X_s is geometrically cellular.

Proposition 8.7. *Let* Y *be a separated scheme of finite type over* $\operatorname{Spec} \mathbb{F}_q$. *If* Y *is geometrically cellular, then* $X \in \mathcal{L}(\mathbb{Z})$*, and in particular the conjecture* $\mathbf{L}^c(Y_{\acute{e}t})$ *holds.*

Proof. Let $k = \mathbb{F}_q$. By the assumption, there exists a finite Galois extension k'/k that gives a cellular decomposition

$$(Y_{k'})^{red} = Y_N \supseteq Y_{N-1} \supseteq \cdots \supseteq Y_{-1} = \varnothing,$$

where $Y_i \setminus Y_{i-1} \cong \mathbb{A}_{k'}^{r_i}$. Affine spaces $\mathbb{A}_{k'}^{r_i}$ lie in $\mathcal{L}(\mathbb{Z})$, as follows from $\operatorname{Spec} k' \in \mathcal{L}(\mathbb{Z})$ and operation $\mathcal{L}3$). Now by induction, using $\mathcal{L}1$), we conclude that $(Y_{k'})^{red} \in \mathcal{L}(\mathbb{Z})$. Similarly, $\mathcal{L}1$) implies that $Y_{k'} \in \mathcal{L}(\mathbb{Z})$. Aplying $\mathcal{L}4$) to the finite étale Galois cover $Y_{k'} \to Y$, we conclude that $Y \in \mathcal{L}(\mathbb{Z})$. ☐

Finally, it is proved in [Mor2014, Proposition 5.14] that if $X \to \operatorname{Spec} \mathcal{O}_F$ is a flat, separated scheme of finite type over the ring of integers of a number field, and X is geometrically cellular, then $X \in \mathcal{L}(\mathbb{Z})$. In particular, the conjecture $\mathbf{L}^c(X_{\acute{e}t})$ holds for such X.

A Some homological algebra

This appendix collects some basic results about the derived category of abelian groups $\mathbf{D}(\mathbb{Z})$ that are used throughout the text. The lemmas below are essentially isolated from [FM2018], with some modifications to deal with 2-torsion.

First we recall that every complex of abelian groups $A^\bullet$ is quasi-isomorphic to its cohomology:

$$A^\bullet \cong \bigoplus_{i\in\mathbb{Z}} H^i(A^\bullet)[-i] \cong \prod_{i\in\mathbb{Z}} H^i(A^\bullet)[-i] = \Big(\cdots \to H^{i-1}(A^\bullet) \xrightarrow{0} H^i(A^\bullet) \xrightarrow{0} H^{i+1}(A^\bullet) \to \cdots\Big).$$

This gives us a useful expression for morphisms in the derived category. Since

$$\operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(A, B[i]) = \begin{cases} \operatorname{Hom}_{\mathbb{Z}}(A,B), & i = 0, \\ \operatorname{Ext}^1_{\mathbb{Z}}(A,B), & i = 1, \\ 0, & \text{otherwise}, \end{cases}$$

we obtain

$$\begin{aligned} \operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet, B^\bullet) &\cong \operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(\bigoplus_{i\in\mathbb{Z}} H^i(A^\bullet)[-i], \prod_{j\in\mathbb{Z}} H^j(B^\bullet)[-j]) \\ &\cong \prod_{i\in\mathbb{Z}}\prod_{j\in\mathbb{Z}} \operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(H^i(A^\bullet), H^j(B^\bullet)[i-j]) \\ &\cong \prod_{i\in\mathbb{Z}} \big(\operatorname{Hom}_{\mathbb{Z}}(H^i(A^\bullet), H^i(B^\bullet)) \oplus \operatorname{Ext}^1_{\mathbb{Z}}(H^i(A^\bullet), H^{i-1}(B^\bullet))\big). \end{aligned}$$

Let us write down this formula for further reference:

$$\operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet, B^\bullet) \cong \prod_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathbb{Z}}(H^i(A^\bullet), H^i(B^\bullet)) \oplus \prod_{i\in\mathbb{Z}} \operatorname{Ext}^1_{\mathbb{Z}}(H^i(A^\bullet), H^{i-1}(B^\bullet)). \tag{A.1}$$

Lemma A.1.

1) *If $C^\bullet$ and $C'^\bullet$ are almost perfect in the sense of Definition 1.1, then the group $\operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(C^\bullet, C'^\bullet)$ has no nontrivial divisible subgroups.*

2) *If $A^\bullet$ is a complex such that $H^i(A^\bullet)$ are finite dimensional $\mathbb{Q}$-vector spaces and $C^\bullet$ is a complex such that $H^i(C^\bullet)$ are finitely generated abelian groups, then the group $\operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet, C^\bullet)$ is divisible.*

Proof. In 1), we consider the decomposition (A.1) for $\operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(C^\bullet, C'^\bullet)$, and observe that under our assumptions, both groups $\prod_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathbb{Z}}(H^i(C^\bullet), H^i(C'^\bullet))$ and $\prod_{i\in\mathbb{Z}} \operatorname{Ext}^1_{\mathbb{Z}}(H^i(C^\bullet), H^{i-1}(C'^\bullet))$ will be of the form $G \oplus T$, where G is a finitely generated abelian group and T is 2-torsion. From this we see that if $x \in \operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(C^\bullet, C'^\bullet)$ is divisible by all powers of 2, then $x = 0$.

Similarly, in part 2), we consider the decomposition (A.1) for $\operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet, C^\bullet)$. By our assumptions $\operatorname{Hom}_{\mathbb{Z}}(H^i(A^\bullet), H^i(C^\bullet)) = 0$ for all i, and each $\operatorname{Ext}^1_{\mathbb{Z}}(H^i(A^\bullet), H^{i-1}(C^\bullet))$ is a direct sum of finitely many groups isomorphic to $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$, which is divisible. Therefore, $\operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet, C^\bullet)$ is a direct product of divisible groups, hence divisible. ∎

Recall that Verdier's axiom (TR1) tells us that every morphism $v\colon A^\bullet \to B^\bullet$ may be completed to a distinguished triangle $A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \xrightarrow{w} A^\bullet[1]$. The axiom (TR3) tells that for every commutative diagram with distinguished rows

$$\begin{array}{ccccccc} A^\bullet & \xrightarrow{u} & B^\bullet & \xrightarrow{v} & C^\bullet & \xrightarrow{w} & A^\bullet[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \\ A'^\bullet & \xrightarrow{u'} & B'^\bullet & \xrightarrow{v'} & C'^\bullet & \xrightarrow{w'} & A'^\bullet[1] \end{array} \tag{A.2}$$

there exists some $h\colon C^\bullet \to C'^\bullet$ giving a morphism of distinguished triangles

$$\begin{array}{ccccccc}
A^\bullet & \xrightarrow{u} & B^\bullet & \xrightarrow{v} & C^\bullet & \xrightarrow{w} & A^\bullet[1] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \exists h} & & \downarrow{\scriptstyle f[1]} \\
A'^\bullet & \xrightarrow{u'} & B'^\bullet & \xrightarrow{v'} & C'^\bullet & \xrightarrow{w'} & A'^\bullet[1]
\end{array} \tag{A.3}$$

The cone $C^\bullet$ in (TR1) and the morphism h in (TR3) are neither unique nor canonical. Two different cones of the same morphism are necessarily isomorphic, but the isomorphism between them is not unique, because it is provided by (TR3). This is a notorious issue with the derived category formalism. Let us recall a useful standard argument which shows that at least in some special cases, things are uniquely defined. The following is basically [BBD1982, Proposition 1.1.9, Corollaire 1.1.10].

Lemma A.2. *Consider the derived category $\mathbf{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$.*

1) For a commutative diagram (A.2), assume that the homomorphism of abelian groups

$$w^*\colon \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(A^\bullet[1], C'^\bullet) \to \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(C^\bullet, C'^\bullet)$$

induced by w is trivial. Then there exists a unique morphism $h\colon C^\bullet \to C'^\bullet$ giving a morphism of triangles (A.3).

2) For a distinguished triangle $A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \xrightarrow{w} A^\bullet[1]$, assume that for any other cone $C'^\bullet$ of u the morphism w^* is trivial. Then in fact the cone of u is unique up to a unique isomorphism.

Proof. In 1), applying $\operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(-, C'^\bullet)$ to the first distinguished triangle, we obtain an exact sequence of abelian groups

$$\operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(A^\bullet[1], C'^\bullet) \xrightarrow{w^*} \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(C^\bullet, C'^\bullet) \xrightarrow{v^*} \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(B^\bullet, C'^\bullet).$$

If $w^* = 0$, we conclude that v^* is a monomorphism. This means that there is a unique morphism h such that $h \circ v = v' \circ g$. Now in 2), if $C^\bullet$ and $C'^\bullet$ are two different cones of u, we have a commutative diagram

$$\begin{array}{ccccccc}
A^\bullet & \xrightarrow{u} & B^\bullet & \xrightarrow{v} & C^\bullet & \xrightarrow{w} & A^\bullet[1] \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow{\scriptstyle id} \\
A^\bullet & \xrightarrow{u'} & B^\bullet & \xrightarrow{v'} & C'^\bullet & \xrightarrow{w'} & A^\bullet[1]
\end{array}$$

As always, by the "triangulated 5-lemma", the dashed arrow is an isomorphism, and it is unique thanks to part 1). □

Here is a particular case that we are going to use.

Corollary A.3. *Consider the derived category $\mathbf{D}(\mathbb{Z})$.*

1) Suppose we have a commutative diagram with distinguished rows (A.2), where $A^\bullet$ is a complex such that $H^i(A^\bullet)$ are finite dimensional $\mathbb{Q}$-vector spaces and $C^\bullet$ and $C'^\bullet$ are almost perfect complexes in the sense of Definition 1.1. Then there exists a unique morphism $h\colon C^\bullet \to C'^\bullet$ giving a morphism of triangles (A.3).

2) For a distinguished triangle

$$A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \xrightarrow{w} A^\bullet[1]$$

assume that $A^\bullet$ is a complex such that $H^i(A^\bullet)$ are finite dimensional $\mathbb{Q}$-vector spaces and $C^\bullet$ is an almost perfect complex. Then the cone of u is unique up to a unique isomorphism.

Proof. In this situation, according to A.1, the group $\operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(C^\bullet, C'^\bullet)$ has no nontrivial divisible subgroups, and $\operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet[1], C'^\bullet)$ is divisible. This means that there are no nontrivial homomorphisms $\operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet[1], C'^\bullet) \to \operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(C^\bullet, C'^\bullet)$, and we may apply A.2. □

Lemma A.4. *Suppose that $A^\bullet$ and $B^\bullet$ are almost of cofinite type in the sense of Definition 1.1. Then a morphism $f\colon A^\bullet \to B^\bullet$ is torsion in $\mathbf{D}(\mathbb{Z})$ (i.e. a torsion element in the group $\operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet, B^\bullet)$, i.e. $f \otimes \mathbb{Q} = 0$) if and only if the morphisms $H^i(f)\colon H^i(A^\bullet) \to H^i(B^\bullet)$ are torsion; that is, they are trivial on the maximal divisible subgroups:*

$$(H^i(f)_{div}\colon H^i(A^\bullet)_{div} \to H^i(B^\bullet)_{div}) = 0.$$

Proof. In the formula (A.1) the groups $H^i(A^\bullet)$ and $H^{i-1}(B^\bullet)$ are of the form $(\mathbb{Q}/\mathbb{Z})^{\oplus r} \oplus T$, where T is finite, and we calculate that

$$\operatorname{Ext}^1_{\mathbb{Z}}((\mathbb{Q}/\mathbb{Z})^{\oplus r} \oplus T, (\mathbb{Q}/\mathbb{Z})^{\oplus r'} \oplus T') \cong T'^{\oplus r} \oplus \operatorname{Ext}^1_{\mathbb{Z}}(T, T')$$

are finite groups.

For $i \gg 0$, the groups $H^i(A^\bullet)$ and $H^{i-1}(B^\bullet)$ will be finite 2-torsion, and therefore $\operatorname{Ext}^1_{\mathbb{Z}}(H^i(A^\bullet), H^{i-1}(B^\bullet))$ will be finite 2-torsion as well. It follows that the whole product $\prod_{i\in\mathbb{Z}} \operatorname{Ext}^1_{\mathbb{Z}}(H^i(A^\bullet), H^{i-1}(B^\bullet))$ is of the form $G \oplus T$, where G is finite and T is possibly infinite 2-torsion. We have therefore $(G \oplus T) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.

Similarly, the group $\prod_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathbb{Z}}(H^i(A^\bullet), H^i(B^\bullet))$ will consist of some part of the form $\widehat{\mathbb{Z}}^{\oplus r} \oplus G$, where G is finite, and some 2-torsion part, which is killed by tensoring with $\mathbb{Q}$. It follows that there is an isomorphism

$$\begin{gathered}\operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet, B^\bullet) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \prod_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathbb{Z}}(H^i(A^\bullet), H^i(B^\bullet)) \otimes_{\mathbb{Z}} \mathbb{Q},\\ f \otimes \mathbb{Q} \mapsto (H^i(f) \otimes \mathbb{Q})_{i\in\mathbb{Z}}.\end{gathered}$$ □

Lemma A.5. *If $A^\bullet$ is a complex of $\mathbb{Q}$-vector spaces and $B^\bullet$ is a complex almost of cofinite type in the sense of Definition 1.1, then there is an isomorphism of abelian groups*

$$\begin{gathered}\operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet, B^\bullet) \xrightarrow{\cong} \prod_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathbb{Z}}(H^i(A^\bullet), H^i(B^\bullet)),\\ f \mapsto (H^i(f))_{i\in\mathbb{Z}}.\end{gathered}$$

Proof. In the formula (A.1), if $H^i(A^\bullet)$ are $\mathbb{Q}$-vector spaces and $H^{i-1}(B^\bullet)$ have form $(\mathbb{Q}/\mathbb{Z})^{\oplus r} \oplus T$ with T finite, we see that the summand with $\operatorname{Ext}^1_{\mathbb{Z}}(H^i(A^\bullet), H^{i-1}(B^\bullet))$ vanishes. □

B Cohomology with compact support

Let us first recall the definition of étale cohomology with compact support. For any arithmetic scheme $f\colon X \to \operatorname{Spec}\mathbb{Z}$ (separated, of finite type) there exists a **Nagata compactification**

$$\begin{array}{ccc} X & \overset{j}{\hookrightarrow} & \mathfrak{X} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & \operatorname{Spec}\mathbb{Z} & \end{array}$$

where j is an open immersion and g is a proper morphism. This is a result of Nagata, and a modern exposition (following Deligne) may be found in [Con2007, Con2009]. See also [SGA 4, Exposé XVII].

Definition B.1. Let X be an arithmetic scheme and let $\mathcal{F}$ an abelian torsion sheaf on $X_{\acute{e}t}$. Then one defines the **cohomology with compact support of** $\mathcal{F}$ via the complex

$$R\Gamma_c(X_{\acute{e}t}, \mathcal{F}) := R\Gamma(\mathfrak{X}_{\acute{e}t}, j_!\mathcal{F}). \tag{B.1}$$

For torsion sheaves, this does not depend on the choice of $j\colon X \hookrightarrow \mathfrak{X}$, but here we would like to fix this choice to be able to compare cohomology with compact support on $X_{\acute{e}t}$ with the singular cohomology with compact support on $X(\mathbb{C})$.

Comparison with analytic cohomology

Definition B.2. Given a Nagata compactification $j\colon X \hookrightarrow \mathfrak{X}$, we consider the corresponding open immersion $j(\mathbb{C})\colon X(\mathbb{C}) \to \mathfrak{X}(\mathbb{C})$, and for a sheaf $\mathcal{F}$ on $X(\mathbb{C})$ we define

$$\Gamma_c(X(\mathbb{C}), \mathcal{F}) := \Gamma(\mathfrak{X}(\mathbb{C}), j(\mathbb{C})_!\mathcal{F}).$$

Similarly, for a $G_\mathbb{R}$-equivariant sheaf on $X(\mathbb{C})$ we define

$$\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathcal{F}) := \Gamma(G_\mathbb{R}, \mathfrak{X}(\mathbb{C}), j(\mathbb{C})_!\mathcal{F}).$$

The canonical reference for comparison between étale and singular cohomology is [SGA 4, Exposé XI, §4], so let us to borrow some definitions and notation from there. Let X be an arithmetic scheme (separated, of finite type over $\operatorname{Spec}\mathbb{Z}$).

1. The base change from $\operatorname{Spec}\mathbb{Z}$ to $\operatorname{Spec}\mathbb{C}$ gives us a morphism of sites $\gamma\colon X_{\mathbb{C},\acute{e}t} \to X_{\acute{e}t}$.

2. As always, we denote by $X(\mathbb{C})$ the set of complex points of X equipped with the usual analytic topology. Let X_{cl}[*] be the site of étale maps $f\colon U \to X(\mathbb{C})$. A covering family in X_{cl} is a family of maps $\{U_i \to U\}$ such that U is the union of images of U_i.

   As the inclusion of an open subset $U \subset X(\mathbb{C})$ is trivially an étale map, we have a fully faithful functor $X(\mathbb{C}) \subset X_{cl}$, and the topology on $X(\mathbb{C})$ is induced by the topology on X_{cl}. This gives us a morphism of sites $\delta\colon X_{cl} \to X(\mathbb{C})$, which by the well-known "comparison lemma" [SGA 4, Exposé III, Théorème 4.1] induces an equivalence of the corresponding categories of sheaves $\delta_*\colon \mathbf{Sh}(X_{cl}) \to \mathbf{Sh}(X(\mathbb{C}))$.

3. A morphism of schemes $f\colon X'_\mathbb{C} \to X_\mathbb{C}$ over $\operatorname{Spec}\mathbb{C}$ is étale if and only if $f(\mathbb{C})\colon X'(\mathbb{C}) \to X(\mathbb{C})$ is étale in the topological sense [SGA 1, Exposé XII, Proposition 3.1], and therefore the functor $X'_\mathbb{C} \rightsquigarrow X'(\mathbb{C})$ gives us a morphism of sites $\epsilon\colon X_{cl} \to X_{\mathbb{C},\acute{e}t}$.

[*] "cl" for "classique".

We may now consider the composite functor

$$\mathbf{Sh}(X_{\acute{e}t}) \xrightarrow{\gamma^*} \mathbf{Sh}(X_{\mathbb{C},\acute{e}t}) \xrightarrow{\epsilon^*} \mathbf{Sh}(X_{cl}) \xrightarrow[\simeq]{\delta_*} \mathbf{Sh}(X(\mathbb{C}))$$

where γ^* is given by the base change from $\operatorname{Spec}\mathbb{Z}$ to $\operatorname{Spec}\mathbb{C}$, the functor ϵ^* is the comparison, and δ_* is an equivalence of categories. As we start from a scheme over $\operatorname{Spec}\mathbb{Z}$ and base change to $\operatorname{Spec}\mathbb{C}$, the resulting sheaf on $X(\mathbb{C})$ is in fact equivariant with respect to the complex conjugation, and the above composition gives us an "inverse image" functor

$$\alpha^* \colon \mathbf{Sh}(X_{\acute{e}t}) \to \mathbf{Sh}(G_\mathbb{R}, X(\mathbb{C})).$$

Proposition B.3. *Given an sheaf $\mathcal{F}$ on $X_{\acute{e}t}$, there exists a natural morphism*

$$\Gamma(X_{\acute{e}t}, \mathcal{F}) \to \Gamma(G_\mathbb{R}, X(\mathbb{C}), \alpha^*\mathcal{F}),$$

and similarly for cohomology with compact support,

$$\Gamma_c(X_{\acute{e}t}, \mathcal{F}) \to \Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \alpha^*\mathcal{F}).$$

Proof. This is standard and follows from the functoriality of α^*. As for cohomology with compact support, if $j\colon X \hookrightarrow \mathfrak{X}$ is a Nagata compactification, we have the corresponding compactification $j(\mathbb{C})\colon X(\mathbb{C}) \hookrightarrow \mathfrak{X}(\mathbb{C})$. The extension by zero morphism $j(\mathbb{C})_!\colon \mathbf{Sh}(X(\mathbb{C})) \to \mathbf{Sh}(\mathfrak{X}(\mathbb{C}))$ restricts to the subcategory of $G_\mathbb{R}$-equivariant sheaves: if $\mathcal{F}$ is a $G_\mathbb{R}$-equivariant sheaf on $X(\mathbb{C})$, then $j(\mathbb{C})_!\mathcal{F}$ is a $G_\mathbb{R}$-equivariant sheaf on $\mathfrak{X}(\mathbb{C})$ (this is evident from the definition of equivariant sheaves as equivariant espaces étalés). It should be clear from the definition of α^* that there is a commutative diagram

$$\begin{array}{ccc} \mathbf{Sh}(X_{\acute{e}t}) & \xrightarrow{\alpha^*} & \mathbf{Sh}(G_\mathbb{R}, X(\mathbb{C})) \\ {\scriptstyle j_!}\downarrow & & \downarrow{\scriptstyle j(\mathbb{C})_!} \\ \mathbf{Sh}(\mathfrak{X}_{\acute{e}t}) & \xrightarrow[\alpha^*_\mathfrak{X}]{} & \mathbf{Sh}(G_\mathbb{R}, \mathfrak{X}(\mathbb{C})) \end{array}$$

(For instance, note that this diagram commutes for representable étale sheaves, and then every étale sheaf is a colimit of representable sheaves, and α^*, $j_!$, $\alpha^*_\mathfrak{X}$, $j(\mathbb{C})_!$ preserve colimits, as left adjoints.)

Now the morphism in question is given by

$$\Gamma_c(X_{\acute{e}t}, \mathcal{F}) := \Gamma(\mathfrak{X}_{\acute{e}t}, j_!\mathcal{F}) \to \Gamma(G_\mathbb{R}, \mathfrak{X}(\mathbb{C}), \alpha^*_\mathfrak{X} j_!\mathcal{F}) = \Gamma(G_\mathbb{R}, \mathfrak{X}(\mathbb{C}), j(\mathbb{C})_!\,\alpha^*\mathcal{F}) =: \Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \alpha^*\mathcal{F}). \quad \square$$

The morphism α is also discussed in [FM2018, Appendix A], but Flach and Morin work with proper schemes; the above remarks are to make sure that everything works fine for compactifications.

Modified étale cohomology

Here we briefly review the **modified étale cohomology with compact support** $R\widehat{\Gamma}_c(X_{\acute{e}t}, -)$. It was introduced by Th. Zink in [Hab1978, Appendix 2] for the case of number rings $X = \operatorname{Spec}\mathcal{O}_{K,S}$, and it is also discussed in [Mil2006, §II.2]. The general definition for $X \to \operatorname{Spec}\mathbb{Z}$ is treated in [FM2018, §6.7] and [GS2018, §2].

Note that thanks to the Leray spectral sequence $R\Gamma(\mathfrak{X}_{\acute{e}t}, -) \cong R\Gamma(\operatorname{Spec}\mathbb{Z}_{\acute{e}t}, -) \circ Rg_*$, we have

$$R\Gamma_c(X_{\acute{e}t}, \mathcal{F}) := R\Gamma(\mathfrak{X}_{\acute{e}t}, j_!\mathcal{F}) \cong R\Gamma((\operatorname{Spec}\mathbb{Z})_{\acute{e}t}, Rf_!\mathcal{F}), \quad \text{where } Rf_!\mathcal{F} := Rg_* j_!\mathcal{F}. \tag{B.2}$$

The formulas (B.1) and (B.2) give two equivalent definitions of cohomology with compact support.

First we recall that for a finite group G and a G-module A the corresponding group cohomology $H^i(G, A)$ (resp. Tate cohomology $\widehat{H}^i(G, A)$) may be defined in terms of resolutions $P_\bullet$ (resp. complete resolutions

$\widehat{P}_\bullet$) of $\mathbb{Z}$ by free $\mathbb{Z}G$-modules (see e.g. [Bro1994, Chapter VI]). Slightly more generally, if $A^\bullet$ is a bounded (cohomological) complex of G-modules, we obtain a *double complex* of abelian groups $\operatorname{Hom}^{\bullet\bullet}(P_\bullet, A^\bullet)$ (resp. $\operatorname{Hom}^{\bullet\bullet}(\widehat{P}_\bullet, A^\bullet)$), and it makes sense to define the corresponding **group hypercohomology** (resp. **Tate hypercohomology**) via the complexes

$$R\Gamma(G, A^\bullet) := \operatorname{Tot}^\oplus(\operatorname{Hom}^{\bullet\bullet}(P_\bullet, A^\bullet)), \quad R\widehat{\Gamma}(G, A^\bullet) := \operatorname{Tot}^\oplus(\operatorname{Hom}^{\bullet\bullet}(\widehat{P}_\bullet, A^\bullet)).$$

Now if $\mathcal{F}$ is an abelian sheaf on $(\operatorname{Spec}\mathbb{Z})_{\acute{e}t}$, then the corresponding **modified cohomology with compact support** is characterized by the distinguished triangle

$$R\widehat{\Gamma}_c((\operatorname{Spec}\mathbb{Z})_{\acute{e}t}, \mathcal{F}) \to R\Gamma((\operatorname{Spec}\mathbb{Z})_{\acute{e}t}, \mathcal{F}) \to R\widehat{\Gamma}(G_\mathbb{R}, v^*\mathcal{F}) \to R\widehat{\Gamma}_c((\operatorname{Spec}\mathbb{Z})_{\acute{e}t}, \mathcal{F})[1]$$

Here $v\colon \operatorname{Spec}\mathbb{R} \to \operatorname{Spec}\mathbb{Z}$ is the canonical morphism, and $v^*\mathcal{F}$ is the corresponding sheaf on $(\operatorname{Spec}\mathbb{R})_{\acute{e}t}$, which may be viewed as a $G_\mathbb{R}$-module by [SGA 4, Exposé VII, 2.3], and $R\widehat{\Gamma}(G_\mathbb{R}, v^*\mathcal{F})$ denotes the corresponding Tate cohomology.

In general, given an arithmetic scheme $X \to \operatorname{Spec}\mathbb{Z}$ and a torsion abelian sheaf $\mathcal{F}$ on $X_{\acute{e}t}$, we pick a Nagata compactification

$$\begin{array}{ccc} X & \overset{j}{\hookrightarrow} & \mathfrak{X} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & \operatorname{Spec}\mathbb{Z} & \end{array}$$

and set

$$R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathcal{F}) := R\widehat{\Gamma}_c((\operatorname{Spec}\mathbb{Z})_{\acute{e}t}, Rf_!\mathcal{F}).$$

We have a natural morphism

$$R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathcal{F}) \to R\Gamma_c(X_{\acute{e}t}, \mathcal{F}),$$

which is an isomorphism if $X(\mathbb{R}) = \varnothing$. In general, there is an exact sequence

$$\cdots \to \widehat{H}^{i-1}(G_\mathbb{R}, (v^*Rf_!\mathcal{F})) \to \widehat{H}^i_c(X_{\acute{e}t}, \mathcal{F}) \to H^i_c(X_{\acute{e}t}, \mathcal{F}) \to \widehat{H}^i(G_\mathbb{R}, (v^*Rf_!\mathcal{F})) \to \cdots$$

where the groups $\widehat{H}^i(G_\mathbb{R}, (v^*Rf_!\mathcal{F}))$ are annihilated by multiplication by $2 = \#G_\mathbb{R}$, which means that $\widehat{H}^i_c(X_{\acute{e}t}, \mathcal{F}) \to H^i_c(X_{\acute{e}t}, \mathcal{F})$ has 2-torsion kernel and cokernel.

For canonicity and functoriality, I refer to [GS2018, §2].

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