Weil-étale cohomology for arbitrary arithmetic schemes and $n < 0$. Part II: The special value conjecture

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Abstract

Following the ideas of Flach and Morin [FM2018], we state a conjecture in terms of Weil-étale cohomology for the vanishing order and special value of the zeta function $\zeta(X, s)$ at $s = n < 0$, where X is a separated scheme of finite type over $\operatorname{Spec} \mathbb{Z}$. We prove that the conjecture is compatible with closed-open decompositions of schemes and affine bundles, and as a consequence, that it holds for cellular schemes over certain 1-dimensional bases.

This is a continuation of author's preprint [Bes2020], which gives a construction of Weil-étale cohomology for $n < 0$ under the mentioned assumptions on X.

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1 Introduction

Let X be an **arithmetic scheme**, by which we will mean throughout this paper that it is a separated scheme of finite type over $\operatorname{Spec} \mathbb{Z}$. Then the corresponding zeta function is defined by

$$\zeta(X, s) = \prod_{\substack{x \in X \\ \text{closed pt.}}} \frac{1}{1 - \#\kappa(x)^{-s}},$$

where $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ denotes the residue field of a point. The above product converges for $\operatorname{Re} s > \dim X$, and conjecturally admits a meromorphic continuation to the whole complex plane. For basic facts and conjectures about zeta functions of schemes, see Serre's survey [Ser1965].

Of particular interest are the so-called special values of $\zeta(X,s)$ at integers $s = n \in \mathbb{Z}$. To define these, assume that $\zeta(X,s)$ admits a meromorphic continuation around $s = n$. We will denote by $d_n = \operatorname{ord}_{s=n}\zeta(X,s)$ the **vanishing order** of $\zeta(X,s)$ at $s = n$. That is, $d_n > 0$ (resp. $d_n < 0$) if $\zeta(X,s)$ has a zero (resp. pole) of order d_n at $s = n$. The corresponding **special value** of $\zeta(X,s)$ at $s = n$ is defined to be the leading nonzero coefficient of the Taylor expansion:

$$\zeta^*(X,n) = \lim_{s\to n}(s-n)^{-d_n}\,\zeta(X,s).$$

Early on Stephen Lichtenbaum conjectured that both numbers $\operatorname{ord}_{s=n}\zeta(X,s)$ and $\zeta^*(X,n)$ should have a cohomological interpretation, related to étale motivic cohomology of X (see e.g. [Lic1984] for varieties over finite fields). This is made more precise in Lichtenbaum's **Weil-étale program**. In particular it suggests the existence of a cohomology theory $H^i_{W,c}(X,\mathbb{Z}(n))$, **Weil-étale cohomology** with compact support, which encodes the information about the vanishing order and special value of $\zeta(X,n)$ at $s = n$.

For Lichtenbaum's recent work on the topic, we refer to his papers [Lic2005, Lic2009b, Lic2009a, Lic2021]. The case of varieties over finite fields $X/\mathbb{F}_q$ was further studied by Thomas Geisser [Gei2004b, Gei2006, Gei2010a], and it is rather well understood now.

Matthias Flach and Baptiste Morin considered the case of proper, regular arithmetic schemes X. In [FM2012] they defined and studied the corresponding Weil-étale topos. Later, Morin gave in [Mor2014] an explicit construction of Weil-étale cohomology groups $H^i_{W,c}(X,\mathbb{Z}(n))$ for $n = 0$, and X a proper, regular arithmetic scheme, under assumptions on finite generation of suitable étale motivic cohomology groups. This construction was further generalized by Flach and Morin in [FM2018] to any $n \in \mathbb{Z}$, again for proper and regular X.

Motivated by the work of Flach and Morin, the author constructed in [Bes2020] the Weil-étale cohomology groups $H^i_{W,c}(X,\mathbb{Z}(n))$ for any arithmetic scheme X (thus removing the assumptions that X is proper or regular) and $n < 0$ (which apparently simplifies certain aspects of the theory). The construction relies on the following assumption (see [Bes2020, §8] for further details and known cases).

Conjecture. $\mathbf{L}^c(X_{\acute{e}t},n)$: the cohomology groups $H^i(X_{\acute{e}t},\mathbb{Z}^c(n))$ are finitely generated for all $i \in \mathbb{Z}$.

Namely, assuming $\mathbf{L}^c(X_{\acute{e}t},n)$, we defined in [Bes2020] perfect complexes of abelian groups $R\Gamma_{W,c}(X,\mathbb{Z}(n))$. This text is a continuation of [Bes2020] and it explores the conjectural relation of our Weil-étale cohomology to the special value of $\zeta(X,s)$ at $s = n < 0$. Specifically, we make the following conjectures.

1) **Conjecture VO**(X,n): *the vanishing order is given by*

$$\operatorname{ord}_{s=n}\zeta(X,s) = \sum_{i\in\mathbb{Z}}(-1)^i\cdot i\cdot \operatorname{rk}_{\mathbb{Z}} H^i_{W,c}(X,\mathbb{Z}(n)).$$

2) A consequence of **Conjecture B**(X,n) (see §2): *after tensoring with* $\mathbb{R}$, *one obtains a long exact sequence of finite dimensional real vector spaces*

$$\cdots \to H^{i-1}_{W,c}(X,\mathbb{R}(n)) \xrightarrow{\smile\theta} H^i_{W,c}(X,\mathbb{R}(n)) \xrightarrow{\smile\theta} H^{i+1}_{W,c}(X,\mathbb{R}(n)) \to \cdots$$

Here $H^i_{W,c}(X,\mathbb{R}(n)) = H^i_{W,c}(X,\mathbb{Z}(n))\otimes_{\mathbb{Z}}\mathbb{R} = H^i(R\Gamma_{W,c}(X,\mathbb{Z}(n))\otimes_{\mathbb{Z}}\mathbb{R})$.

It follows that there is a canonical isomorphism

$$\lambda\colon \mathbb{R} \xrightarrow{\cong} (\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)))\otimes\mathbb{R}.$$

Here $\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n))$ is the determinant of the perfect complex of abelian groups $R\Gamma_{W,c}(X,\mathbb{Z}(n))$, in the sense of Knudsen and Mumford [KM1976]. In particular, $\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n))$ is a free $\mathbb{Z}$-module of rank 1. For the reader's convenience, we include a brief overview of determinants in appendix A.

3) **Conjecture C**(X,n): *the special value is determined up to sign by*

$$\lambda(\zeta^*(X,n)^{-1})\cdot\mathbb{Z} = \det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)).$$

If X is proper and regular, then our construction of $R\Gamma_{W,c}(X,\mathbb{Z}(n))$ and the above conjectures agree with those of Flach and Morin from [FM2018]. Apart from removing the assumption that X is proper and regular, one novelty of this work is that we prove the compatibility of the above conjectures with operations on schemes, in particular closed-open decompositions $Z \not\hookrightarrow X \hookleftarrow U$, where $Z \subset X$ is a closed subscheme and $U = X \setminus Z$ is the open complement, and affine bundles $\mathbb{A}^r_X = \mathbb{A}^r_{\mathbb{Z}} \times X$. (See proposition 6.3 and theorem 6.8.) This gives a machinery that allows one to start from the particular instances of schemes for which the conjectures are known, and construct new schemes for which the conjectures hold as well. As an application, we prove in §7 the following result.

Main theorem. *Let B be a 1-dimensional arithmetic scheme, such that each of the generic points $\eta \in B$ satisfies one of the following properties:*

a) $\operatorname{char} \kappa(\eta) = p > 0$*;*

b) $\operatorname{char} \kappa(\eta) = 0$*, and* $\kappa(\eta)/\mathbb{Q}$ *is an abelian number field.*

If X is a B-cellular arithmetic scheme with smooth quasi-projective fiber $X_{red,\mathbb{C}}$, then the conjectures $\mathbf{VO}(X,n)$ *and* $\mathbf{C}(X,n)$ *hold unconditionally for any $n < 0$.*

In fact, this result will be established for a bigger class of arithmetic schemes $\mathcal{C}(\mathbb{Z})$; we refer to §7 for further details.

Outline of the paper

In §2 we define the regulator morphism, based on the construction of Kerr, Lewis, and Müller-Stach [KLMS2006], and state the conjecture $\mathbf{B}(X,n)$ related to it.

Then §3 is devoted to the vanishing order conjecture $\mathbf{VO}(X,n)$. We also explain why it is consistent with a conjecture of Soulé and the vanishing orders that come from the expected functional equation

In §4 we state the special value conjecture $\mathbf{C}(X,n)$.

We explain in §5 that for $X/\mathbb{F}_q$ a variety over a finite field, the special value conjecture $\mathbf{C}(X,n)$ is consistent with the conjectures considered by Geisser in [Gei2004b, Gei2006, Gei2010a]. In particular, it holds for any smooth projective $X/\mathbb{F}_q$, assuming the conjecture $\mathbf{L}^c(X_{ét},n)$. This also generalizes to all varieties $X/\mathbb{F}_q$ if we assume the resolution of singularities over $\mathbb{F}_q$.

Then we prove in §6 that the conjectures $\mathbf{VO}(X,n)$ and $\mathbf{C}(X,n)$ are compatible with basic operations on schemes: disjoint unions, closed-open decompositions, and affine bundles. Using these results, we conclude in §7 with a class of schemes for which the special value conjecture holds unconditionally.

For convenience of the reader, the appendix A briefly reviews basic definitions and facts related to the determinants of complexes that are relevant to this text.

Notation

In this paper, X will always denote an **arithmetic scheme** (separated, of finite type over $\operatorname{Spec}\mathbb{Z}$), and n will always be a strictly negative integer.

We denote by $R\Gamma_{fg}(X,\mathbb{Z}(n))$ and $R\Gamma_{W,c}(X,\mathbb{Z}(n))$ the complexes constructed in [Bes2020], assuming the conjecture $\mathbf{L}^c(X_{ét},n)$ stated above. For $\mathbb{Q}$-coefficients, we put

$$R\Gamma_{fg}(X,\mathbb{Q}(n)) := R\Gamma_{fg}(X,\mathbb{Z}(n)) \otimes^{\mathbf{L}}_{\mathbb{Z}} \mathbb{Q} = R\Gamma_{fg}(X,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q},$$
$$R\Gamma_{W,c}(X,\mathbb{Q}(n)) := R\Gamma_{W,c}(X,\mathbb{Z}(n)) \otimes^{\mathbf{L}}_{\mathbb{Z}} \mathbb{Q} = R\Gamma_{W,c}(X,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Accordingly,

$$H^i_{fg}(X,\mathbb{Q}(n)) := H^i(R\Gamma_{fg}(X,\mathbb{Q}(n))) = H^i_{fg}(X,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q},$$
$$H^i_{W,c}(X,\mathbb{Q}(n)) := H^i(R\Gamma_{W,c}(X,\mathbb{Q}(n))) = H^i_{W,c}(X,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Similarly, we define the corresponding complexes and cohomology with $\mathbb{R}$-coefficients.

By $X(\mathbb{C})$ we denote the space of complex points of X with the usual analytic topology. It carries a natural action of $G_\mathbb{R} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ via complex conjugation. For a subring $A \subseteq \mathbb{R}$ we denote by $A(n)$ the $G_\mathbb{R}$-module $(2\pi i)^n A$, and also the corresponding constant $G_\mathbb{R}$-equivariant sheaf on $X(\mathbb{C})$.

We denote by $R\Gamma_c(X(\mathbb{C}), A(n))$ (resp. $R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), A(n)))$ the cohomology with compact support (resp. $G_\mathbb{R}$-equivariant cohomology with compact support) of X with $A(n)$-coefficients. For more details on $G_\mathbb{R}$-equivariant cohomology, see [Bes2020]. With real coefficients, $H^i_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n)) = H^i_c(X(\mathbb{C}), \mathbb{R}(n))^{G_\mathbb{R}}$, where the $G_\mathbb{R}$-action on $H^i_c(X(\mathbb{C}), \mathbb{R}(n))$ naturally comes from the corresponding action on $X(\mathbb{C})$ and $\mathbb{R}(n)$. The **Borel–Moore homology** is defined as dual to the cohomology with compact support. In particular, we will be interested in

$$R\Gamma_{BM}(X(\mathbb{C}), \mathbb{R}(n)) := R\mathrm{Hom}(R\Gamma_c(X(\mathbb{C}), \mathbb{R}(n)), \mathbb{R}),$$
$$R\Gamma_{BM}(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n)) := R\mathrm{Hom}(R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n)), \mathbb{R}).$$

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2 Regulator morphism

To state the special value conjecture, we need to define the regulator morphism from motivic cohomology to Deligne-Beilinson (co)homology. It was introduced by Bloch in [Blo1986b], and here we are going to use the construction of Kerr, Lewis, and Müller-Stach from [KLMS2006], which works on the level of complexes. We will simply call it "the KLM morphism". It works under assumption that $X_{red,\mathbb{C}}$ is a smooth quasi-projective variety.

For simplicity we will assume in this section that X is reduced (the motivic cohomology does not distinguish between X and X_{red}), and that $X_\mathbb{C}$ is connected of dimension $d_\mathbb{C}$ (otherwise, the arguments below may be applied to each connected component). We fix a compactification by a normal crossing divisor

$$X_\mathbb{C} \xhookrightarrow{j} \overline{X}_\mathbb{C} \hookleftarrow D$$

The KLM regulator takes form of a morphism in the derived category

$$z^p(X_\mathbb{C}, -\bullet) \otimes \mathbb{Q} \to {}'C^{2p-2d_\mathbb{C}+\bullet}_\mathcal{D}(\overline{X}_\mathbb{C}, D, \mathbb{Q}(p - d_\mathbb{C})). \tag{2.1}$$

Here $z^p(X_\mathbb{C}, -\bullet)$ denotes the Bloch's cycle complex [Blo1986a]. We recall that by the definition, $z^p(X_\mathbb{C}, i)$ is freely generated by algebraic cycles $Z \subset X_\mathbb{C} \times_{\mathrm{Spec}\,\mathbb{C}} \Delta^i_\mathbb{C}$ of codimension p that intersect properly the faces of the algebraic simplex $\Delta^i_\mathbb{C} = \mathrm{Spec}\,\mathbb{C}[t_0, \ldots, t_i]/(1 - \sum_j t_j)$. For us it will be more convenient to pass to

$$z_{d_\mathbb{C}-p}(X_\mathbb{C}, i) = z^p(X_\mathbb{C}, i),$$

generated by cycles $Z \subset X_\mathbb{C} \times_{\mathrm{Spec}\,\mathbb{C}} \Delta^i_\mathbb{C}$ of dimension $p + i$.

The complex ${}'C^\bullet_\mathcal{D}(\overline{X}_\mathbb{C}, D, \mathbb{Q}(k))$ on the right hand side of (2.1) computes Deligne(–Beilinson) homology, as defined by Jannsen [Jan1988]. In particular, taking $p = d_\mathbb{C} + 1 - n$, tensoring with $\mathbb{R}$, and shifting by $2n$, we obtain

$$z_{n-1}(X_\mathbb{C}, -\bullet) \otimes \mathbb{R}[2n] \to {}'C^{2+\bullet}_\mathcal{D}(\overline{X}_\mathbb{C}, D, \mathbb{R}(1 - n)). \tag{2.2}$$

2.1. Remark. Some comments are in order.

1. Originally, the KLM morphism is defined using a cubical version of cycle complexes, but these are quasi-isomorphic to the usual simplicial cycle complexes (see [Lev1994]), so we do not make a distinction here. For a simplicial version, see also [KLL2018].
2. The KLM morphism is defined as a genuine morphism of complexes (not just a morphism in the derived category) on a certain subcomplex $z^r_{\mathbb{R}}(X_{\mathbb{C}}, \bullet) \subset z^r(X_{\mathbb{C}}, \bullet)$. This inclusion is a quasi-isomorphism, if we pass rational coefficients. This is stated without tensoring with $\mathbb{Q}$ in the original paper [KLMS2006], but the omission is acknowledged later in [KL2007]. For our particular purposes, it will be enough to have a regulator with coefficients in $\mathbb{Q}$, or in fact in $\mathbb{R}$.
3. The case of smooth quasi-projective $X_{\mathbb{C}}$, where one has to consider a compactification by a normal crossing divisor as above, is treated in [KLMS2006, §5.9].

Now we make a little digression to identify the right hand side of (2.2). Under our assumption that $n < 0$, the Deligne homology corresponds to Borel–Moore homology.

2.2. Lemma. *For any $n < 0$ there is a quasi-isomorphism*

$$'C^\bullet_{\mathcal{D}}(\overline{X_{\mathbb{C}}}, D, \mathbb{R}(1-n)) \cong R\Gamma_{BM}(X(\mathbb{C}), \mathbb{R}(n))[-1] := R\mathrm{Hom}(R\Gamma_c(X(\mathbb{C}), \mathbb{R}(n)), \mathbb{R})[-1].$$

Further, this respects the natural actions of $G_{\mathbb{R}}$ on both complexes.

Proof. From the proof of [Jan1988, Theorem 1.15], for any $k \in \mathbb{Z}$ we have a quasi-isomorphism

$$'C^\bullet_{\mathcal{D}}(\overline{X_{\mathbb{C}}}, D, \mathbb{R}(k)) \cong R\Gamma(\overline{X}(\mathbb{C}), \mathbb{R}(k + d_{\mathbb{C}})_{\mathcal{D}\text{-B},(\overline{X}_{\mathbb{C}}, X_{\mathbb{C}})})[2d_{\mathbb{C}}], \tag{2.3}$$

where

$$\mathbb{R}(k + d_{\mathbb{C}})_{\mathcal{D}\text{-B},(\overline{X}_{\mathbb{C}}, X_{\mathbb{C}})} = \mathrm{Cone}\Big(Rj_*\mathbb{R}(k + d_{\mathbb{C}}) \oplus \Omega^{\geqslant k+d_{\mathbb{C}}}_{\overline{X}(\mathbb{C})}(\log D) \xrightarrow{\epsilon - \iota} Rj_*\Omega^\bullet_{X(\mathbb{C})}\Big)[-1]$$

is the sheaf whose hypercohomology on $\overline{X}(\mathbb{C})$ gives Deligne-Beilinson cohomology (see [EV1988] for further details).

Here $\Omega^\bullet_{\overline{X}(\mathbb{C})}$ denotes the usual de Rham complex of holomorphic differential forms, and $\Omega^\bullet_{\overline{X}(\mathbb{C})}(\log D)$ is the complex of forms with at most logarithmic poles along $D(\mathbb{C})$. The latter complex is filtered by subcomplexes $\Omega^{\geqslant\bullet}_{\overline{X}(\mathbb{C})}(\log D)$. The morphism $\epsilon\colon Rj_*\mathbb{R}(k) \to Rj_*\Omega^\bullet_{X(\mathbb{C})}$ is induced by the canonical morphism of sheaves $\mathbb{R}(k) \to \mathcal{O}_{X(\mathbb{C})}$, and ι is induced by the natural inclusion $\Omega^\bullet_{\overline{X}(\mathbb{C})}(\log D) \xrightarrow{\cong} j_*\Omega^\bullet_{X(\mathbb{C})} = Rj_*\Omega^\bullet_{X(\mathbb{C})}$, which is a quasi-isomorphism of filtered complexes.

We will be interested in the case of $k > 0$, when the part $\Omega^{\geqslant k+d_{\mathbb{C}}}_{\overline{X}(\mathbb{C})}(\log D)$ disappears, and we obtain

$$\begin{aligned}
\mathbb{R}(k + d_{\mathbb{C}})_{\mathcal{D}\text{-B},(\overline{X}_{\mathbb{C}}, X_{\mathbb{C}})} &\cong Rj_*\,\mathrm{Cone}\Big(\mathbb{R}(k + d_{\mathbb{C}}) \xrightarrow{\epsilon} \Omega^\bullet_{X(\mathbb{C})}\Big)[-1] \\
&\cong Rj_*\Big(\mathbb{R}(k + d_{\mathbb{C}}) \xrightarrow{\epsilon} \Omega^\bullet_{X(\mathbb{C})}[-1]\Big) \\
&\cong Rj_*\Big(\mathbb{R}(k + d_{\mathbb{C}}) \to \mathbb{C}[-1]\Big) \qquad (2.4) \\
&\cong Rj_*\mathbb{R}(k + d_{\mathbb{C}} - 1)[-1] \qquad (2.5)
\end{aligned}$$

Here (2.4) comes from the Poincaré lemma $\mathbb{C} \cong \Omega^\bullet_{X(\mathbb{C})}$, and (2.5) comes from the short exact sequence of $G_{\mathbb{R}}$-modules $\mathbb{R}(k + d_{\mathbb{C}}) \rightarrowtail \mathbb{C} \twoheadrightarrow \mathbb{R}(k + d_{\mathbb{C}} - 1)$.

Coming back to (2.3) for $k = 1 - n$, we conclude that

$$\begin{aligned}
'C^\bullet_{\mathcal{D}}(\overline{X_{\mathbb{C}}}, D, \mathbb{R}(1-n)) &\cong R\Gamma(X(\mathbb{C}), \mathbb{R}(d_{\mathbb{C}} - n))[2d_{\mathbb{C}} - 1] \\
&\cong R\mathrm{Hom}(R\Gamma_c(X(\mathbb{C}), \mathbb{R}(n)), \mathbb{R})[-1].
\end{aligned}$$

Here the last isomorphism is Poincaré duality. □

Now coming back to (2.2), the above lemma allows us to reinterpret the KLM morphism as

$$z_{n-1}(X_{\mathbb{C}}, -\bullet) \otimes \mathbb{R}[2n] \to R\Gamma_{BM}(X(\mathbb{C}), \mathbb{R}(n)), \mathbb{R})[1]. \tag{2.6}$$

By the definition, we have

$$z_{n-1}(X_{\mathbb{C}}, -\bullet) \otimes \mathbb{R}[2n] = z_{n-1}(X_{\mathbb{C}}, -\bullet) \otimes \mathbb{R}[2n-2][2] = \Gamma(X_{\mathbb{C},\acute{e}t}, \mathbb{R}^c(n-1))[2], \tag{2.7}$$

where the complex of sheaves $\mathbb{R}^c(p)$ is defined by $U \rightsquigarrow z_p(U, -\bullet) \otimes \mathbb{R}[2p]$. By étale cohomological descent [Gei2010b, Theorem 3.1], we have*

$$\Gamma(X_{\mathbb{C},\acute{e}t}, \mathbb{R}^c(n-1)) \cong R\Gamma(X_{\mathbb{C},\acute{e}t}, \mathbb{R}^c(n-1)). \tag{2.8}$$

Finally, the base change from X to $X_{\mathbb{C}}$ naturally maps cycles $Z \subset X \times \Delta^i_{\mathbb{Z}}$ of dimension n to cycles in $X_{\mathbb{C}} \times_{\operatorname{Spec}\mathbb{C}} \Delta^i_{\mathbb{C}}$ of dimension $n-1$, so that there is a morphism

$$R\Gamma(X_{\acute{e}t}, \mathbb{R}^c(n)) \to R\Gamma(X_{\mathbb{C},\acute{e}t}, \mathbb{R}^c(n-1))[2]. \tag{2.9}$$

2.3. Remark. Assuming that X is flat of pure Krull dimension d, we have $\mathbb{R}^c(n)^X = \mathbb{R}(d-n)^X[2d]$, where $\mathbb{R}(\bullet)$ is the usual cycle complex defined by $z^n(\text{␣}, -\bullet)[-2n]$. Similarly, $\mathbb{R}^c(n)^{X_{\mathbb{C}}} = \mathbb{R}(d_{\mathbb{C}} - n)^{X_{\mathbb{C}}}[2d_{\mathbb{C}}]$, with $d_{\mathbb{C}} = d-1$. With this renumbering, the morphism (2.9) becomes

$$R\Gamma(X_{\acute{e}t}, \mathbb{R}(d-n))[2d] \to R\Gamma(X_{\mathbb{C},\acute{e}t}, \mathbb{R}(d-n))[2d].$$

This probably looks more natural, but we do not impose extra assumptions on X and work exclusively with complexes $A^c(\bullet)$ defined in terms of dimension of algebraic cycles, instead of $A(\bullet)$ defined in terms of codimension.

2.4. Definition. Given an arithmetic scheme X with smooth quasi-projective $X_{\mathbb{C}}$, and $n < 0$, consider the composition of morphisms

$$\begin{aligned} R\Gamma(X_{\acute{e}t}, \mathbb{R}^c(n)) \xrightarrow{(2.9)} R\Gamma(X_{\mathbb{C},\acute{e}t}, \mathbb{R}^c(n-1))[2] &\overset{(2.8)}{\cong} \Gamma(X_{\mathbb{C},\acute{e}t}, \mathbb{R}^c(n-1))[2] \\ &\overset{(2.7)}{=} z_{n-1}(X_{\mathbb{C}}, -\bullet)_{\mathbb{R}}[2n] \xrightarrow{(2.6)} R\Gamma_{BM}(X(\mathbb{C}), \mathbb{R}(n)), \mathbb{R})[1]. \end{aligned}$$

Further, we take $G_{\mathbb{R}}$-invariants, which gives us the **(étale) regulator**

$$Reg_{X,n} \colon R\Gamma(X_{\acute{e}t}, \mathbb{R}^c(n)) \to R\Gamma_{BM}(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[1].$$

Now we state our conjecture about the regulator, which will play an important role in everything that follows.

2.5. Conjecture. $\mathbf{B}(X,n)$: given an arithmetic scheme X with smooth quasi-projective $X_{\mathbb{C}}$ and $n < 0$, the regulator morphism $Reg_{X,n}$ induces a quasi-isomorphism of complexes of real vector spaces

$$Reg^{\vee}_{X,n} \colon R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \to R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}).$$

2.6. Remark. If $X/\mathbb{F}_q$ is a variety over a finite field, then $X(\mathbb{C}) = \varnothing$, so the regulator map in not interesting. Indeed, in our setting, its purpose is to take care of the archimedian places of X. In this case $\mathbf{B}(X,n)$ implies that $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are torsion groups. However, by [Bes2020, Proposition 4.2], the finite generation conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$ already implies that $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are finite groups.

2.7. Remark. We reiterate that our construction of $Reg_{X,n}$ works for $X_{red,\mathbb{C}}$ smooth quasi-projective. In everything that follows, whenever the regulator morphism or the conjecture $\mathbf{B}(X,n)$ is brought, we will tacitly assume this. This is quite unfortunate, because the Weil-étale complexes $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ were constructed in [Bes2020] for any arithmetic scheme, assuming only $\mathbf{L}^c(X_{\acute{e}t}, n)$. Defining the regulator for singular $X_{red,\mathbb{C}}$ is an interesting project for further work.

*We note that [Gei2010b, Theorem 3.1] holds unconditionally, since the Beilinson–Lichtenbaum conjecture follows from the Bloch–Kato conjecture, which is now a theorem. See also [Gei2004a] where the consequences of Bloch–Kato for motivic cohomology are deduced.

3 Vanishing order conjecture

Assuming that $\zeta(X,s)$ admits a meromorphic continuation around $s = n < 0$, we state the following conjecture for the vanishing order at $s = n$.

3.1. Conjecture. $\mathbf{VO}(X,n)$: one has

$$\operatorname{ord}_{s=n} \zeta(X,s) = \chi'(R\Gamma_{W,c}(X,\mathbb{Z}(n))) := \sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot \operatorname{rk}_{\mathbb{Z}} H^i_{W,c}(X,\mathbb{Z}(n)).$$

We note that the right hand side makes sense assuming the conjecture $\mathbf{L}^c(X_{\acute{e}t},n)$, under which $H^i_{W,c}(X,\mathbb{Z}(n))$ are finitely generated groups, trivial for $|i| \gg 0$ (see [Bes2020, Proposition 7.12]).

3.2. Remark. The conjecture $\mathbf{VO}(X,n)$ is similar to [FM2018, Conjecture 5.11]. When X is proper and regular, then $\mathbf{VO}(X,n)$ is the same as the vanishing order conjecture of Flach and Morin. Namely, the latter takes form

$$\operatorname{ord}_{s=n} \zeta(X,s) = \sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot \dim_{\mathbb{R}} H^i_{ar,c}(X,\widetilde{\mathbb{R}}(n)), \tag{3.1}$$

where by definition,

$$R\Gamma_{ar,c}(X,\widetilde{\mathbb{R}}(n)) := R\Gamma_c(X,\mathbb{R}(n)) \oplus R\Gamma_c(X,\mathbb{R}(n))[-1].$$

Moreover, [FM2018, Proposition 4.14], gives a distinguished triangle

$$R\Gamma_{dR}(X_{\mathbb{R}}/\mathbb{R})/\operatorname{Fil}^n[-2] \to R\Gamma_{ar,c}(X,\widetilde{\mathbb{R}}(n)) \to R\Gamma_{W,c}(X,\mathbb{Z}(n)) \otimes \mathbb{R} \to R\Gamma_{dR}(X_{\mathbb{R}}/\mathbb{R})/\operatorname{Fil}^n[-1]$$

Therefore, in case of $n < 0$, we have $R\Gamma_{ar,c}(X,\widetilde{\mathbb{R}}(n)) \cong R\Gamma_{W,c}(X,\mathbb{Z}(n)) \otimes \mathbb{R}$, and (3.1) is exactly our conjecture $\mathbf{VO}(X,n)$.

3.3. Remark. The alternating sum in the formula is the so-called **secondary Euler characteristic** of the Weil-étale complex $R\Gamma_{W,c}(X,\mathbb{Z}(n))$. The easy calculations below show that the usual Euler characteristic of $R\Gamma_{W,c}(X,\mathbb{Z}(n))$ vanishes, assuming conjectures $\mathbf{L}^c(X_{\acute{e}t},n)$ and $\mathbf{B}(X,n)$. See [Ram2016] for more details about secondary Euler characteristic and its appearances in nature.

Under the regulator conjecture, our conjectural vanishing order formula takes form of the usual Euler characteristic of the equivariant cohomology $R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{R}(n))$ or motivic cohomology $R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n))[1]$.

3.4. Proposition. *Assuming* $\mathbf{L}^c(X_{\acute{e}t},n)$ *and* $\mathbf{B}(X,n)$*, the conjecture* $\mathbf{VO}(X,n)$ *is equivalent to*

$$\begin{aligned} \operatorname{ord}_{s=n} \zeta(X,s) &= \chi(R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{R}(n)) = \sum_{i\in\mathbb{Z}} (-1)^i \dim_{\mathbb{R}} H^i_c(X(\mathbb{C}),\mathbb{R}(n))^{G_{\mathbb{R}}} \\ &= -\chi(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n))) = \sum_{i\in\mathbb{Z}} (-1)^{i+1} \operatorname{rk}_{\mathbb{Z}} H^i(X_{\acute{e}t},\mathbb{Z}^c(n)). \end{aligned}$$

Further, we have

$$\chi(R\Gamma_{W,c}(X,\mathbb{Z}(n))) = 0.$$

Proof. Thanks to [Bes2020, Proposition 7.14], the Weil-étale complex tensored with $\mathbb{R}$ splits as

$$R\Gamma_{W,c}(X,\mathbb{R}(n)) \cong R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R})[-1] \oplus R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{R}(n))[-1].$$

Assuming the conjecture $\mathbf{B}(X,n)$, we also have a quasi-isomorphism

$$R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{R}(n))[-1] \cong R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R}),$$

so that

$$\dim_{\mathbb{R}} H^i_{W,c}(X,\mathbb{R}(n)) = \dim_{\mathbb{R}} H^{i-1}_c(X(\mathbb{C}),\mathbb{R}(n))^{G_{\mathbb{R}}} + \dim_{\mathbb{R}} H^{i-2}_c(X(\mathbb{C}),\mathbb{R}(n))^{G_{\mathbb{R}}}.$$

Using this, we may rewrite the sum

$$\begin{aligned}\sum_{i\in\mathbb{Z}}(-1)^i\cdot i\cdot \operatorname{rk}_{\mathbb{Z}} H^i_{W,c}(X,\mathbb{Z}(n)) &= \sum_{i\in\mathbb{Z}}(-1)^i\cdot i\cdot \dim_{\mathbb{R}} H^i_{W,c}(X,\mathbb{R}(n))\\ &= \sum_{i\in\mathbb{Z}}(-1)^i\cdot i\cdot \dim_{\mathbb{R}} H^{i-1}_c(X(\mathbb{C}),\mathbb{R}(n))^{G_{\mathbb{R}}} + \sum_{i\in\mathbb{Z}}(-1)^i\cdot i\cdot \dim_{\mathbb{R}} H^{i-2}_c(X(\mathbb{C}),\mathbb{R}(n))^{G_{\mathbb{R}}}\\ &= -\sum_{i\in\mathbb{Z}}(-1)^i \dim_{\mathbb{R}} H^{i-1}_c(X(\mathbb{C}),\mathbb{R}(n))^{G_{\mathbb{R}}} = \chi(R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{R}(n)).\end{aligned}$$

Similarly,

$$\sum_{i\in\mathbb{Z}}(-1)^i\cdot i\cdot \operatorname{rk}_{\mathbb{Z}} H^i_{W,c}(X,\mathbb{Z}(n)) = \chi(R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R})[1]) = -\chi(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n))).$$

These considerations also show that the usual Euler characteristic of $R\Gamma_{W,c}(X,\mathbb{Z}(n))$ vanishes. □

3.5. Remark. The conjecture $\mathbf{VO}(X,n)$ is related to a conjecture of Soulé [Sou1984, Conjecture 2.2], which originally reads in terms of K'-theory

$$\operatorname{ord}_{s=n}\zeta(X,s) = \sum_{i\in\mathbb{Z}}(-1)^{i+1}\dim_{\mathbb{Q}} K'_i(X)_{(n)}.$$

Then, as explained in [Kah2005, Remark 43], this may be rewritten in terms of Borel–Moore motivic homology as

$$\sum_{i\in\mathbb{Z}}(-1)^{i+1}\dim_{\mathbb{Q}} H^{BM}_i(X,\mathbb{Q}(n)).$$

In our setting, $R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n))$ plays the role of Borel–Moore homology, which explains the formula

$$\operatorname{ord}_{s=n}\zeta(X,s) = \sum_{i\in\mathbb{Z}}(-1)^{i+1}\operatorname{rk}_{\mathbb{Z}} H^i(X_{\acute{e}t},\mathbb{Z}^c(n)).$$

3.6. Remark ([FM2018, Proposition 5.13]). As for the formula

$$\operatorname{ord}_{s=n}\zeta(X,s) = \sum_{i\in\mathbb{Z}}(-1)^i\dim_{\mathbb{R}} H^i_c(X(\mathbb{C}),\mathbb{R}(n))^{G_{\mathbb{R}}},$$

it basically means that the the vanishing order at $s=n<0$ comes from the archimedian Γ-factor that appears in the (hypothetical) functional equation, as explained in [Ser1970, §§3,4] (see also [FM2020, §4]).

Namely, assuming that $X_{\mathbb{C}}$ is a smooth projective variety, we consider the Hodge decomposition

$$H^i(X(\mathbb{C}),\mathbb{C}) = \bigoplus_{p+q=i} H^{p,q},$$

which carries an action of $G_{\mathbb{R}}=\{id,\sigma\}$ such that $\sigma(H^{p,q})=H^{q,p}$. We set $h^{p,q}=\dim_{\mathbb{C}} H^{p,q}$. For $p=i/2$ consider the eigenspace decomposition $H^{p,p}=H^{p,+}\oplus H^{p,-}$, where

$$\begin{aligned}H^{p,+} &= \{x\in H^{p,p}\mid \sigma(x)=(-1)^p\,x\},\\ H^{p,-} &= \{x\in H^{p,p}\mid \sigma(x)=(-1)^{p+1}\,x\},\end{aligned}$$

and set accordingly $h^{p,\pm}=\dim_{\mathbb{C}} H^{p,\pm}$. It is expected that the completed zeta function

$$\zeta(\overline{X},s) = \zeta(X,s)\,\zeta(X_\infty,s),$$

satisfies a functional equation of the form

$$A^{\frac{d-s}{2}}\,\zeta(\overline{X}, d-s) = A^{\frac{s}{2}}\,\zeta(\overline{X}, s).$$

Here

$$\zeta(X_\infty, s) = \prod_{i\in\mathbb{Z}} L_\infty(H^i(X), s)^{(-1)^i},$$

$$L_\infty(H^i(X), s) = \prod_{p=i/2} \Gamma_\mathbb{R}(s-p)^{h^{p,+}}\,\Gamma_\mathbb{R}(s-p+1)^{h^{p,-}} \prod_{\substack{p+q=i\\ p<q}} \Gamma_\mathbb{C}(s-p)^{h^{p,q}},$$

$$\Gamma_\mathbb{R}(s) = \pi^{-s/2}\,\Gamma(s/2), \quad \Gamma_\mathbb{C}(s) = (2\pi)^{-s}\,\Gamma(s).$$

Therefore, the expected vanishing order at $s = n < 0$ is

$$\begin{aligned}\operatorname{ord}_{s=n}\zeta(X,s) = -\operatorname{ord}_{s=n}\zeta(X_\infty,s) &= -\sum_{i\in\mathbb{Z}}(-1)^i \operatorname{ord}_{s=n} L_\infty(H^i(X),s)\\ &= \sum_{i\in\mathbb{Z}}(-1)^i\Bigl(\sum_{p=i/2} h^{p,(-1)^{n-p}} + \sum_{\substack{p+q=i\\ p<q}} h^{p,q}\Bigr).\end{aligned}$$

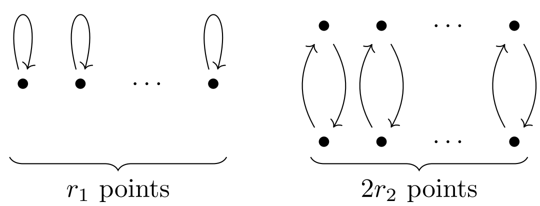
The last equality comes from the fact that $\Gamma(s)$ has simple poles at all $s = n \le 0$. We have

$$\begin{aligned}\dim_\mathbb{R} H^i(X(\mathbb{C}),\mathbb{R}(n))^{G_\mathbb{R}} = \dim_\mathbb{R} H^i(X(\mathbb{C}),\mathbb{R})^{\sigma=(-1)^n} &= \dim_\mathbb{C} H^i(X(\mathbb{C}),\mathbb{C})^{\sigma=(-1)^n}\\ &= \sum_{p=i/2} h^{p,(-1)^{n-p}} + \sum_{\substack{p+q=i\\ p<q}} h^{p,q}.\end{aligned}$$

Here the terms $h^{p,q}$ with $p < q$ come from $\sigma(H^{p,q}) = H^{q,p}$, while $h^{p,(-1)^{n-p}}$ come from the action on $H^{p,p}$. We see that our conjectural formula gives the expected vanishing orders.

Let us see some particular examples when the meromorphic continuation for $\zeta(X,s)$ is known.

3.7. Example. Suppose that $X = \operatorname{Spec}\mathcal{O}_F$ is the spectrum of the ring of integers of a number field $F/\mathbb{Q}$. Let r_1 be the number of real embeddings $F \hookrightarrow \mathbb{R}$ and r_2 the number of conjugate pairs of complex embeddings $F \hookrightarrow \mathbb{C}$. The space $X(\mathbb{C})$ with the action of complex conjugation may be pictured as follows:



The complex $R\Gamma_c(X(\mathbb{C}),\mathbb{R}(n))$ consists of a single $G_\mathbb{R}$-module in degree 0 given by

$$\mathbb{R}(n)^{\oplus r_1} \oplus (\mathbb{R}(n)\oplus\mathbb{R}(n))^{\oplus r_2},$$

with the action of $G_\mathbb{R}$ on the first summand $\mathbb{R}(n)^{\oplus r_1}$ by complex conjugation and the action on the second summand $(\mathbb{R}(n)\oplus\mathbb{R}(n))^{\oplus r_2}$ via $(x,y)\mapsto(\overline{y},\overline{x})$. The corresponding real space of fixed points has dimension

$$\dim_\mathbb{R} H^0_c(G_\mathbb{R}, X(\mathbb{C}),\mathbb{R}(n)) = \begin{cases} r_2, & n \text{ odd},\\ r_1+r_2, & n \text{ even},\end{cases}$$

which indeed agrees with the vanishing order of the Dedekind zeta function $\zeta(X,s) = \zeta_F(s)$ at $s = n < 0$.

On the side of motivic cohomology (see e.g. [Gei2017, Proposition 4.14]), given $n < 0$, the groups $H^i_{ét}(X, \mathbb{Z}^c(n)) = H^{i+2}_{ét}(X, \mathbb{Z}(1-n))$ are finite, except for $i = -1$, where

$$\operatorname{rk}_{\mathbb{Z}} H^{-1}_{ét}(X, \mathbb{Z}^c(n)) = \operatorname{rk}_{\mathbb{Z}} H^1_{ét}(X, \mathbb{Z}(1-n)) = \begin{cases} r_2, & n \text{ odd}, \\ r_1 + r_2, & n \text{ even}. \end{cases}$$

3.8. Example. Suppose that X is a variety over a finite field $\mathbb{F}_q$. Then the vanishing order conjecture is not very interesting, as the formula gives

$$\operatorname{ord}_{s=n} \zeta(X,s) = \sum_{i\in\mathbb{Z}} (-1)^i \dim_{\mathbb{R}} H^i_c(X(\mathbb{C}), \mathbb{R}(n))^{G_{\mathbb{R}}} = \sum_{i\in\mathbb{Z}} (-1)^{i+1} \operatorname{rk}_{\mathbb{Z}} H^i(X_{ét}, \mathbb{Z}^c(n)) = 0,$$

since $X(\mathbb{C}) = \varnothing$, and also because $\mathbf{L}^c(X_{ét}, n)$ implies $\operatorname{rk}_{\mathbb{Z}} H^i(X_{ét}, \mathbb{Z}^c(n)) = 0$ in case of varieties over finite fields [Bes2020, Proposition 4.2]. Therefore, the conjecture simply asserts that $\zeta(X,s)$ does not have zeros or poles at $s = n < 0$.

This is indeed the case. We have $\zeta(X,s) = Z(X, q^{-s})$, where

$$Z(X,t) = \exp\Bigl(\sum_{k\ge 1} \frac{\#X(\mathbb{F}_{q^k})}{k}\, t^k\Bigr)$$

is the Hasse–Weil zeta function. By Deligne's work on Weil's conjectures [Del1980], the zeros and poles of $Z(X,s)$ satisfy $|s| = q^{-w/2}$, where $0 \le w \le 2\dim X$ (see e.g. [Kat1994, pp. 26–27]). In particular, q^{-s} for $s = n < 0$ is not a zero or pole of $Z(X,s)$.

We also note that our definition of $H^i_{W,c}(X, \mathbb{Z}(n))$, and pretty much all the above, makes sense only for $n < 0$. Already for $n = 0$, for instance, the zeta function of a smooth projective curve $X/\mathbb{F}_q$ has a simple pole at $s = 0$.

3.9. Example. Let $X = E$ be an integral model of an elliptic curve over $\mathbb{Q}$. Then as a consequence of the modularity theorem (Wiles–Breuil–Conrad–Diamond–Taylor) it is known that $\zeta(E,s)$ admits a meromorphic continuation, which satisfies the functional equation with the Γ-factors discussed in 3.6. In this particular case $\operatorname{ord}_{s=n} \zeta(E,s) = 0$ for all $n < 0$. This is consistent with the fact that $\chi(R\Gamma_c(G_{\mathbb{R}}, E(\mathbb{C}), \mathbb{R}(n))) = 0$.

Indeed, the equivariant cohomology groups $H^i_c(E(\mathbb{C}), \mathbb{R}(n))^{G_{\mathbb{R}}}$ are the following:

	$i = 0$	$i = 1$	$i = 2$
n even:	$\mathbb{R}$	$\mathbb{R}$	0
n odd:	0	$\mathbb{R}$	$\mathbb{R}$

—see for instance the calculation in [Sie2019, Lemma A.6].

4 Special value conjecture

4.1. Definition. Define a morphism of complexes

$$\smile\theta\colon R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \to R\Gamma_{W,c}(X, \mathbb{Z}(n))[1] \otimes \mathbb{R}$$

using the splitting

$$R\Gamma_{W,c}(X, \mathbb{R}(n)) \cong R\operatorname{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \oplus R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1]$$

as follows:

$$\begin{tikzcd}
R\Gamma_{W,c}(X,\mathbb{R}(n)) \arrow[r, dashed, "\smile\theta"] \arrow[d, "\cong"] & R\Gamma_{W,c}(X,\mathbb{R}(n))[1] \arrow[d, "\cong"] \\
\begin{array}{c} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R})[-1] \\ \oplus \\ R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{R}(n))[-1] \end{array} \arrow[r, "Reg^\vee_{X,n}"] & \begin{array}{c} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R}) \\ \oplus \\ R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{R}(n)) \end{array}
\end{tikzcd}$$

4.2. Lemma. *Assuming $\mathbf{L}^c(X_{\acute{e}t},n)$ and $\mathbf{B}(X,n)$, the morphism $\smile\theta$ induces a long exact sequence of finite dimensional real vector spaces*

$$\cdots \to H^{i-1}_{W,c}(X,\mathbb{R}(n)) \xrightarrow{\smile\theta} H^i_{W,c}(X,\mathbb{R}(n)) \xrightarrow{\smile\theta} H^{i+1}_{W,c}(X,\mathbb{R}(n)) \to \cdots$$

Proof. We obtain a sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H^{i-1}_{W,c}(X,\mathbb{R}(n)) \arrow[r, dashed, "\smile\theta"] \arrow[d, "\cong"] & H^{i}_{W,c}(X,\mathbb{R}(n)) \arrow[r, dashed, "\smile\theta"] \arrow[d, "\cong"] & H^{i+1}_{W,c}(X,\mathbb{R}(n)) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots & \begin{array}{c} \mathrm{Hom}(H^{-i}(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R}) \\ \oplus \\ H^{i-2}_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{R}(n)) \end{array} \arrow[r, "\cong"] & \begin{array}{c} \mathrm{Hom}(H^{-i-1}(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R}) \\ \oplus \\ H^{i-1}_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{R}(n)) \end{array} \arrow[r, "\cong"] & \begin{array}{c} \mathrm{Hom}(H^{-i-2}(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R}) \\ \oplus \\ H^{i}_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{R}(n)) \end{array} & \cdots
\end{tikzcd}$$

Here the diagonal arrows are isomorphisms according to $\mathbf{B}(X,n)$, so the sequence is exact. □

We recall that the Weil-étale complex $R\Gamma_{W,c}(X,\mathbb{Z}(n))$ was defined in [Bes2020] up to a *non-unique* isomorphism in the derived category $\mathbf{D}(\mathbb{Z})$ via a distinguished triangle

$$R\Gamma_{W,c}(X,\mathbb{Z}(n)) \to R\Gamma_{fg}(X,\mathbb{Z}(n)) \xrightarrow{i_\infty} R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{Z}(n)) \to R\Gamma_{W,c}(X,\mathbb{Z}(n))[1] \tag{4.1}$$

This is quite unpleasant, and there ought to be a better, more canonical construction of $R\Gamma_{W,c}(X,\mathbb{Z}(n))$. However, this is not a big issue for the moment, since the special value conjecture will be formulated not in terms of $R\Gamma_{W,c}(X,\mathbb{Z}(n))$, but in terms of its determinant $\det_\mathbb{Z} R\Gamma_{W,c}(X,\mathbb{Z}(n))$ (see appendix A), which is well-defined.

4.3. Lemma. *The determinant $\det_\mathbb{Z} R\Gamma_{W,c}(X,\mathbb{Z}(n))$ is well-defined up to a canonical isomorphism.*

Proof. Two different choices of the mapping fiber in (4.1) give us an isomorphism of distinguished triangles

$$\begin{tikzcd}
R\Gamma_{W,c}(X,\mathbb{Z}(n)) \arrow[r] \arrow[d, "f", "\cong"'] & R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[r, "i_\infty"] \arrow[d, "id"] & R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{Z}(n)) \arrow[r] \arrow[d, "id"] & R\Gamma_{W,c}(X,\mathbb{Z}(n))[1] \arrow[d, "f", "\cong"'] \\
R\Gamma_{W,c}(X,\mathbb{Z}(n))' \arrow[r] & R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[r, "i_\infty"] & R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{Z}(n)) \arrow[r] & R\Gamma_{W,c}(X,\mathbb{Z}(n))'[1]
\end{tikzcd}$$

Now the idea is to use the functoriality of determinants with respect to isomorphisms of distinguished triangles (see A.4). The only technical issue is that whenever $X(\mathbb{R}) \neq \varnothing$, the complexes $R\Gamma_{fg}(X,\mathbb{Z}(n))$ and $R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{Z}(n))$ are not perfect, but possibly have finite 2-torsion in $H^i(-)$ for arbitrarily big i (in [Bes2020] we called such complexes **almost perfect**). On the other hand, the determinants are only defined for perfect complexes. Luckily, $H^i(i^*_\infty)$ is an isomorphism for $i \gg 0$, so that for m big enough we may take the corresponding canonical truncations $\tau_{\leq m}$:

$$\begin{tikzcd}
\tau_{\leq m}R\Gamma_{W,c}(X,\mathbb{Z}(n)) \arrow[r] \arrow[d, "\cong"] & \tau_{\leq m}R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[r] \arrow[d] & \tau_{\leq m}R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \arrow[r] \arrow[d] & \tau_{\leq m}R\Gamma_{W,c}(X,\mathbb{Z}(n))[1] \arrow[d, "\cong"] \\
R\Gamma_{W,c}(X,\mathbb{Z}(n)) \arrow[r] \arrow[d] & R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[r, "i_\infty"] \arrow[d] & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \arrow[r] \arrow[d] & R\Gamma_{W,c}(X,\mathbb{Z}(n))[1] \arrow[d] \\
0 \arrow[r] \arrow[d] & \tau_{\geq m+1}R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[r, "\cong"] \arrow[d] & \tau_{\geq m+1}R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \arrow[r] \arrow[d] & 0 \arrow[d] \\
\tau_{\leq m}R\Gamma_{W,c}(X,\mathbb{Z}(n))[1] \arrow[r] & \tau_{\leq m}R\Gamma_{fg}(X,\mathbb{Z}(n))[1] \arrow[r] & \tau_{\leq m}R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n))[1] \arrow[r] & \tau_{\leq m}R\Gamma_{W,c}(X,\mathbb{Z}(n))[2]
\end{tikzcd}$$

The truncations give us (rotating the triangles for convenience)

$$\begin{tikzcd}
\tau_{\leq m}R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n))[-1] \arrow[r] \arrow[d, "id"] & R\Gamma_{W,c}(X,\mathbb{Z}(n)) \arrow[r] \arrow[d, "f", "\cong"'] & \tau_{\leq m}R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[r] \arrow[d, "id"] & \tau_{\leq m}R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \arrow[d, "id"] \\
\tau_{\leq m}R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n))[-1] \arrow[r] & R\Gamma_{W,c}(X,\mathbb{Z}(n))' \arrow[r] & \tau_{\leq m}R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[r] & \tau_{\leq m}R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n))
\end{tikzcd}$$

Now according to A.4, we have a commutative diagram

$$\begin{tikzcd}
\det_{\mathbb{Z}}\tau_{\leq m}R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n))[-1]\otimes_{\mathbb{Z}}\det_{\mathbb{Z}}\tau_{\leq m}R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[r, "i", "\cong"'] \arrow[d, "id"] & \det_{\mathbb{Z}}R\Gamma_{W,c}(X,\mathbb{Z}(n)) \arrow[d, "\det_{\mathbb{Z}}(f)", "\cong"'] \\
\det_{\mathbb{Z}}\tau_{\leq m}R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n))[-1]\otimes_{\mathbb{Z}}\det_{\mathbb{Z}}\tau_{\leq m}R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[r, "i'", "\cong"'] & \det_{\mathbb{Z}}R\Gamma_{W,c}(X,\mathbb{Z}(n))
\end{tikzcd}$$

so that $\det_{\mathbb{Z}}(f)=i'\circ i^{-1}$. $\square$

4.4. Lemma. *The non-canonical splitting*

$$R\Gamma_{W,c}(X,\mathbb{Q}(n))\cong R\mathrm{Hom}(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q})[-1]\oplus R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Q}(n))[-1].$$

Gives a canonical isomorphism of determinants

$$\det_{\mathbb{Q}}R\Gamma_{W,c}(X,\mathbb{Q}(n))\cong\det_{\mathbb{Q}}R\mathrm{Hom}(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q})[-1]\otimes_{\mathbb{Q}}\det_{\mathbb{Q}}R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Q}(n))[-1]$$

Proof. This is similar to the previous lemma; in fact, after tensoring with $\mathbb{Q}$, we obtain perfect complexes of $\mathbb{Q}$-vector spaces, so that the truncations are not needed anymore.

We recall from [Bes2020, Proposition 7.4] that $i_\infty^*\otimes\mathbb{Q}=0$, and thanks to this there is an isomorphism of triangles

$$\begin{tikzcd}
R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Q}(n))[-1] \arrow[r, "id"] \arrow[d] & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Q}(n))[-1] \arrow[d] \\
R\Gamma_{W,c}(X,\mathbb{Q}(n)) \arrow[r, dashed, "f", "\cong"'] \arrow[d] & \begin{matrix} R\mathrm{Hom}(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q})[-1] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Q}(n))[-1] \end{matrix} \arrow[d] \\
R\Gamma_{fg}(X,\mathbb{Q}(n)) \arrow[r, "g\otimes\mathbb{Q}", "\cong"'] \arrow[d] & R\mathrm{Hom}(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q})[-1] \arrow[d] \\
R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Q}(n)) \arrow[r, "id"] & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Q}(n))
\end{tikzcd} \tag{4.2}$$

Here the third horizontal arrow comes from the triangle defining $R\Gamma_{fg}(X,\mathbb{Z}(n))$

$$R\mathrm{Hom}(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q})[-2]\xrightarrow{\alpha_{X,n}}R\Gamma_c(X_{ét},\mathbb{Z}(n))\to R\Gamma_{fg}(X,\mathbb{Z}(n))\xrightarrow{g}R\mathrm{Hom}(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q})[-1]$$

tensored with $\mathbb{Q}$ (see [Bes2020, §5]). The distinguished column on the right of (4.2) is the direct sum

$$\begin{array}{ccc} R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Q}(n))[-1] & & 0 \\ \downarrow{\scriptstyle id} & & \downarrow \\ R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Q}(n))[-1] & \oplus & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q})[-1] \\ \downarrow & & \downarrow{\scriptstyle id} \\ 0 & & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q})[-1] \\ \downarrow & & \downarrow \\ R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Q}(n)) & & 0 \end{array}$$

The splitting isomorphism f in (4.2) is by no means canonical. However, after taking the determinants, we obtain a commutative diagram (see A.4)

$$\begin{tikzcd} \begin{array}{c} \det_\mathbb{Q} R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Q}(n))[-1] \\ \otimes_\mathbb{Q} \\ \det_\mathbb{Q} R\Gamma_{fg}(X, \mathbb{Q}(n)) \end{array} \arrow[r, "i", "\cong"'] \arrow[d, "\cong"', "id\otimes\det_\mathbb{Z}(g\otimes\mathbb{Q})"] & \det_\mathbb{Q} R\Gamma_{W,c}(X, \mathbb{Q}(n)) \arrow[d, "\cong"', "\det_\mathbb{Z}(f)"] \arrow[dl, dashed, "\cong"] \\ \begin{array}{c} \det_\mathbb{Q} R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Q}(n))[-1] \\ \otimes_\mathbb{Q} \\ \det_\mathbb{Q} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q})[-1] \end{array} \arrow[r, "i'", "\cong"'] & \det_\mathbb{Q} \left(\begin{array}{c} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q})[-1] \\ \oplus \\ R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Q}(n))[-1] \end{array} \right) \end{tikzcd}$$

Here the dashed diagonal arrow is the desired canonical isomorphism. $\square$

4.5. Definition. Given an arithmetic scheme X and $n < 0$, assume the conjectures $\mathbf{L}^c(X_{\acute{e}t}, n)$ and $\mathbf{B}(X, n)$. Consider the quasi-isomorphism

$$\begin{array}{c} R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n))[-2] \\ \oplus \\ R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n))[-1] \end{array} \xrightarrow[\cong]{Reg^\vee_{X,n}[-1]\oplus id} \begin{array}{c} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \\ \oplus \\ R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n))[-1] \end{array} \xrightarrow[\cong]{\text{split}} R\Gamma_{W,c}(X, \mathbb{R}(n)) \tag{4.3}$$

Note that the first complex has determinant

$$\det_\mathbb{R} \left(\begin{array}{c} R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n))[-2] \\ \oplus \\ R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n))[-1] \end{array} \right) \cong \begin{array}{c} \det_\mathbb{R} R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n)) \\ \otimes_\mathbb{R} \\ (\det_\mathbb{R} R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n)))^{-1} \end{array} \cong \mathbb{R},$$

and for the last complex, by compatibility with base change, we have a canonical isomorphism

$$\det_\mathbb{R} R\Gamma_{W,c}(X, \mathbb{R}(n)) \cong (\det_\mathbb{Z} R\Gamma_{W,c}(X, \mathbb{Z}(n))) \otimes_\mathbb{Z} \mathbb{R}.$$

Therefore, after taking determinants, the quasi-isomorphism (4.3) induces a canonical isomorphism

$$\lambda = \lambda_{X,n} \colon \mathbb{R} \xrightarrow{\cong} (\det_\mathbb{Z} R\Gamma_{W,c}(X, \mathbb{Z}(n))) \otimes_\mathbb{Z} \mathbb{R}. \tag{4.4}$$

4.6. Remark. An equivalent way to define λ is

$$\lambda \colon \mathbb{R} \xrightarrow{\cong} \bigotimes_{i\in\mathbb{Z}} (\det_\mathbb{R} H^i_{W,c}(X, \mathbb{R}(n)))^{(-1)^i} \xrightarrow{\cong} \Bigl(\bigotimes_{i\in\mathbb{Z}} (\det_\mathbb{Z} H^i_{W,c}(X, \mathbb{Z}(n)))^{(-1)^i} \Bigr) \otimes_\mathbb{Z} \mathbb{R}$$
$$\xrightarrow{\cong} (\det_\mathbb{Z} R\Gamma_{W,c}(X, \mathbb{Z}(n))) \otimes_\mathbb{Z} \mathbb{R}.$$

Here the first isomorphism comes from lemma 4.2.

We are ready to state the main conjecture of this paper. The determinant $\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)))$ is a free $\mathbb{Z}$-module of rank 1 (which does not sound very interesting), but the whole point of the determinant business is that the isomorphism (4.4) embeds it canonically into $\mathbb{R}$. We conjecture that this embedding gives the special value of $\zeta(X, s)$ at $s = n$ in the following sense.

4.7. Conjecture. $\mathbf{C}(X, n)$: let X be an arithmetic scheme and $n < 0$ a strictly negative integer. Assuming $\mathbf{L}^c(X_{\acute{e}t}, n)$, $\mathbf{B}(X, n)$, and meromorphic continuation of $\zeta(X, s)$ around $s = n < 0$, the corresponding special value is determined up to sign by

$$\lambda(\zeta^*(X, n)^{-1}) \cdot \mathbb{Z} = \det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)),$$

where λ is the canonical isomorphism (4.4).

4.8. Remark. This conjecture is similar to [FM2018, Conjecture 5.12]. When X is proper and regular, the above conjecture is the same as the special value conjecture of Flach and Morin, which for $n \in \mathbb{Z}$ reads

$$\lambda_\infty\Big(\zeta^*(X, n)^{-1} \cdot C(X, n) \cdot \mathbb{Z}\Big) = \Delta(X/\mathbb{Z}, n). \tag{4.5}$$

Here the fundamental line $\Delta(X/\mathbb{Z}, n)$ is defined via

$$\Delta(X/\mathbb{Z}, n) := \det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} R\Gamma_{dR}(X/\mathbb{Z})/\operatorname{Fil}^n.$$

If $n < 0$, then $\Delta(X/\mathbb{Z}, n) = \det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n))$. Moreover, $C(X, n)$ is some rational number, defined via $\prod_p |c_p(X, n)|_p$. Here $c_p(X, n) \in \mathbb{Q}_p^\times/\mathbb{Z}_p^\times$ are the "local factors" described in [FM2018, §5.4], but [FM2018, Proposition 5.8] says that if $n \le 0$, then $c_p(X, n) \equiv 1 \pmod{\mathbb{Z}_p^\times}$ for all p. Therefore, $C(X, n) = 1$ in our situation. Finally, the trivialization isomorphism "λ_∞" is defined precisely the same way as our λ. Therefore, (4.5) coincides with our conjecture $\mathbf{C}(X, n)$ for $n < 0$.

Flach and Morin prove that their conjecture is consistent with the Tamagawa number conjecture of Bloch–Kato–Fontaine–Perrin-Riou [FPR1994]; see [FM2018, §5.6] for the details.

4.9. Remark. Some canonical isomorphisms of determinants involve multiplication by ± 1, so there is no surprise that the resulting conjecture is stated up to sign ± 1. However, this is not a big issue, since the sign may be recovered from the (conjectural) functional equation.

5 Case of varieties over finite fields

For varieties over finite fields, our special value conjecture corresponds to the conjectures studied by Geisser in [Gei2004b], [Gei2006], [Gei2010a].

5.1. Proposition. *If $X/\mathbb{F}_q$ is a variety over a finite field, then assuming $\mathbf{L}^c(X_{\acute{e}t}, n)$, the special value conjecture $\mathbf{C}(X, n)$ is equivalent to*

$$\zeta^*(X, n) = \pm \prod_{i \in \mathbb{Z}} |H^i_{W,c}(X, \mathbb{Z}(n))|^{(-1)^i} = \pm \prod_{i \in \mathbb{Z}} |H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))|^{(-1)^i} = \pm \prod_{i \in \mathbb{Z}} |H_i^c(X_{ar}, \mathbb{Z}(n))|^{(-1)^{i+1}}, \tag{5.1}$$

where $H_i^c(X_{ar}, \mathbb{Z}(n))$ are Geisser's arithmetic homology groups defined in [Gei2010a].

Proof. Assuming $\mathbf{L}^c(X_{\acute{e}t}, n)$, thanks to [Bes2020, Proposition 7.7] we have

$$H^i_{W,c}(X, \mathbb{Z}(n)) \cong \operatorname{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Hom}(H^c_{i-1}(X_{ar}, \mathbb{Z}(n)), \mathbb{Q}/\mathbb{Z}).$$

The involved cohomology groups are finite and vanish for $|i| \gg 0$ (see [Bes2020, Proposition 4.2]), and therefore by A.6, the determinant is given by

$$\begin{array}{ccc} \det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)) & \subset & \det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q} \\ \| & & \| \\ \frac{1}{m}\mathbb{Z} & \subset & \mathbb{Q} \end{array}$$

where

$$m = \prod_{i\in\mathbb{Z}} |H^i_{W,c}(X,\mathbb{Z}(n))|^{(-1)^i}. \qquad \square$$

5.2. Remark. Formulas similar to (5.1) were suggested long time ago by Lichtenbaum in [Lic1984].

5.3. Theorem. *Let $X/\mathbb{F}_q$ be a variety over a finite field satisfying $\mathbf{L}^c(X_{\acute{e}t}, n)$ for $n < 0$. Then the conjecture $\mathbf{C}(X,n)$ holds.*

We note that (5.1) is equivalent to the special value formula that appears in [Gei2010a, Theorem 4.5]. The conjecture $\mathbf{P}_0(X)$ in the statement of [Gei2010a, Theorem 4.5] is implied by our conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$ thanks to [Gei2010a, Proposition 4.1]. Geisser's proof eventually reduces to [Mil1986], but for our case of $s = n < 0$, the situation is easier, and we can give a more direct explanation, using earlier results from [BN1978] regarding Grothendieck's trace formula.

Proof. In this case the conjecture reduces to

$$\zeta(X,n) = \prod_{i\in\mathbb{Z}} |H^i(X_{\acute{e}t},\mathbb{Z}^c(n))|^{(-1)^i}.$$

By duality [Bes2020, Theorem I], we have

$$|H^{2-i}(X_{\acute{e}t},\mathbb{Z}^c(n))| = |H^i_c(X_{\acute{e}t},\mathbb{Z}(n))|,$$

where

$$\mathbb{Z}(n) := \bigoplus_{\ell\ne p} \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)[-1] := \bigoplus_{\ell\ne p} \mu_{\ell^\infty}^{\otimes n}[-1] := \bigoplus_{\ell\ne p} \varinjlim_r \mu_{\ell^r}^{\otimes n}[-1].$$

Now $H^i_c(X_{\acute{e}t},\mathbb{Q}_\ell(n)) = 0$ for $n < 0$, and therefore $H^i_c(X_{\acute{e}t},\mathbb{Z}_\ell(n)) \cong H^{i-1}_c(X_{\acute{e}t},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$. This means that our conjectural formula may be written as

$$\zeta(X,n) = \prod_{\ell\ne p}\prod_{i\in\mathbb{Z}} |H^i_c(X_{\acute{e}t},\mathbb{Z}_\ell(n))|^{(-1)^i}.$$

Recall that Grothendieck's trace formula (see [Gro1968] or [SGA $4\frac{1}{2}$, Rapport]) reads

$$Z(X,t) = \prod_{i\in\mathbb{Z}} \det\bigl(1 - tF \bigm| H^i_c(\overline{X},\mathbb{Q}_\ell)\bigr)^{(-1)^{i+1}}.$$

Here $\overline{X} := X \times_{\operatorname{Spec}\mathbb{F}_q} \overline{\mathbb{F}}_q$, and F is the Frobenius acting on $H^i_c(\overline{X},\mathbb{Q}_\ell)$. Substituting $t = q^{-n}$,

$$\zeta(X,n) = \prod_{i\in\mathbb{Z}} \det\bigl(1 - q^{-n}F \bigm| H^i_c(\overline{X},\mathbb{Q}_\ell)\bigr)^{(-1)^{i+1}}.$$

Then following the proof of [BN1978, Theorem (3.1)], we obtain for each $\ell \ne p$

$$|\zeta(X,n)|_\ell = \prod_{i\in\mathbb{Z}} |H^i_c(X,\mathbb{Z}_\ell(n))|^{(-1)^{i+1}}. \tag{5.2}$$

On the other hand, for $n < 0$ we have $|\zeta(X,n)|_p = 1$. This fact may be justified, without assuming that X is smooth or projective, for instance, using the trace formula for rigid cohomology [Ked2006, p. 1446], which gives $Z(X,t) = \prod_i P_i(t)^{(-1)^{i+1}}$, where $P_i(t) \in \mathbb{Z}[t]$ and $P_i(0) = 1$. In particular, $P_i(q^{-n}) \equiv 1 \pmod p$. We conclude that the product formula gives from (5.2)

$$\zeta(X,n) = \prod_{\ell\ne p}\prod_{i\in\mathbb{Z}} |H^i_c(X,\mathbb{Z}_\ell(n))|^{(-1)^i},$$

which is precisely our special value formula. $\square$

5.4. Remark. The fact that $|\zeta(X,n)|_p = 1$ observed in the above argument explains why Weil-étale cohomology ignores the p-primary part in a sense.

Let us consider a couple of particular examples to see how the special value conjecture works. It is to be noted that for a general arithmetic scheme X, calculating the motivic cohomology $H^i(X_{ét}, \mathbb{Z}^c(n))$ (and therefore our Weil-étale cohomology $H^i_{W,c}(X, \mathbb{Z}(n))$) is by no means a trivial task. The finite generation of $H^i(X_{ét}, \mathbb{Z}^c(n))$ is only known for particular cases (see [Bes2020, §8]), and calculating the torsion part, which bears the arithmetic information, is even more difficult. Similarly, an explicit calculation of the regulator map $Reg_{X,n}$ is highly nontrivial. Therefore, for the moment we give some toy examples over finite fields.

5.5. Example. If $X = \operatorname{Spec} \mathbb{F}_q$, then $\zeta(X,s) = \frac{1}{1-q^{-s}}$. In this case for $n < 0$ we obtain

$$H^i(\operatorname{Spec} \mathbb{F}_{q,ét}, \mathbb{Z}^c(n)) = H^i(\operatorname{Spec} \mathbb{F}_{q,ét}, \mathbb{Z}(-n)) = \begin{cases} \mathbb{Z}/(q^{-n}-1), & i = 1, \\ 0, & i \neq 1 \end{cases} \tag{5.3}$$

(see for instance [Gei2017, Example 4.2]). Therefore, the formula (5.1) indeed recovers $\zeta(X,n)$ up to sign.

Similarly, replacing $\operatorname{Spec} \mathbb{F}_q$ with $\operatorname{Spec} \mathbb{F}_{q^m}$, viewed as a variety over $\mathbb{F}_q$, we have $\zeta(\operatorname{Spec} \mathbb{F}_{q^m}, s) = \zeta(\operatorname{Spec} \mathbb{F}_q, ms)$, and (5.3) also changes accordingly.

5.6. Example. Consider $X = \mathbb{P}^1_{\mathbb{F}_q}/(0 \sim 1)$, or equivalently, a nodal cubic. The zeta function is $\zeta(X,s) = \frac{1}{1-q^{1-s}}$. We may calculate the groups $H^i(X_{ét}, \mathbb{Z}^c(n))$ using the blowup square

$$\begin{tikzcd} \operatorname{Spec} \mathbb{F}_q \sqcup \operatorname{Spec} \mathbb{F}_q \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{P}^1_{\mathbb{F}_q} \arrow[d] \\ \operatorname{Spec} \mathbb{F}_q \arrow[r] & X \end{tikzcd}$$

This is similar to [Gei2006, §8, Example 2]. Geisser uses eh-topology and long exact sequences associated to abstract blowup squares [Gei2006, Proposition 3.2]. In our case, the same works, since according to [Bes2020, Theorem I], one has $H^i(X_{ét}, \mathbb{Z}^c(n)) \cong \operatorname{Hom}(H^{2-i}_c(X_{ét}, \mathbb{Z}(n)), \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Z}(n) = \varinjlim_{p \nmid m} \mu_m^{\otimes n}[-1]$, and for such sheaves étale cohomology and eh-cohomology agree by [Gei2006, Theorem 3.6].

Using the projective bundle formula, we calculate from (5.3)

$$H^i(\mathbb{P}^1_{\mathbb{F}_q,ét}, \mathbb{Z}^c(n)) = \begin{cases} \mathbb{Z}/(q^{1-n}-1), & i = -1, \\ \mathbb{Z}/(q^{-n}-1), & i = +1, \\ 0, & i \neq \pm 1. \end{cases}$$

Following the same argument from [Gei2006, §8, Example 2], the short exact sequences

$$0 \to H^i(\mathbb{P}^1_{\mathbb{F}_q,ét}, \mathbb{Z}^c(n)) \to H^i(X_{ét}, \mathbb{Z}^c(n)) \to H^{i+1}((\operatorname{Spec} \mathbb{F}_q)_{ét}, \mathbb{Z}^c(n)) \to 0$$

give us

$$H^i(X_{ét}, \mathbb{Z}^c(n)) = \begin{cases} \mathbb{Z}/(q^{1-n}-1), & i = -1, \\ \mathbb{Z}/(q^{-n}-1), & i = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

The formula (5.1) recovers the correct value $\zeta(X,n)$.

5.7. Example. In general, if $X/\mathbb{F}_q$ is a curve, then the conjecture $\mathbf{L}^c(X_{ét}, n)$ holds; see for instance [Gei2017, Proposition 4.3]. The cohomology $H^i(X_{ét}, \mathbb{Z}^c(n))$ is concentrated in degrees $-1, 0, +1$ by duality and the reasons of cohomological dimension, and the special value formula reads

$$\zeta^*(X,n) = \pm \frac{|H^0(X_{ét}, \mathbb{Z}^c(n))|}{|H^{-1}(X_{ét}, \mathbb{Z}^c(n))| \cdot |H^1(X_{ét}, \mathbb{Z}^c(n))|}.$$

6 Compatibility with operations on schemes

From the definition of $\zeta(X,s)$, the following basic properties follow easily.

1) **Disjoint unions**: if $X = \coprod_{1\le i\le r} X_i$ is a finite disjoint union of arithmetic schemes, then

$$\zeta(X,s) = \prod_{1\le i\le r} \zeta(X_i,s). \tag{6.1}$$

   In particular,

$$\begin{aligned} \operatorname{ord}_{s=n} \zeta(X,s) &= \sum_{1\le i\le r} \operatorname{ord}_{s=n} \zeta(X_i,s),\\ \zeta^*(X,n) &= \prod_{1\le i\le r} \zeta^*(X_i,n). \end{aligned}$$

2) **Closed-open decompositions**: if $Z\subset X$ is a closed subscheme and $U = X\backslash Z$ is its open complement, then we will say that we have a **closed-open decomposition** and write $Z \not\hookrightarrow X \hookleftarrow U$. In this case

$$\zeta(X,s) = \zeta(Z,s)\cdot\zeta(U,s). \tag{6.2}$$

   In particular,

$$\begin{aligned} \operatorname{ord}_{s=n} \zeta(X,s) &= \operatorname{ord}_{s=n} \zeta(Z,s) + \operatorname{ord}_{s=n} \zeta(U,s),\\ \zeta^*(X,n) &= \zeta^*(Z,n)\cdot\zeta^*(U,n). \end{aligned}$$

3) **Affine bundles**: for any $r\ge 0$ the zeta function of the relative affine space $\mathbb{A}^r_X = \mathbb{A}^r_{\mathbb{Z}}\times X$ satisfies

$$\zeta(\mathbb{A}^r_X,s) = \zeta(X,s-r). \tag{6.3}$$

   In particular,

$$\begin{aligned} \operatorname{ord}_{s=n} \zeta(\mathbb{A}^r_X,s) &= \operatorname{ord}_{s=n-r} \zeta(X,s),\\ \zeta^*(\mathbb{A}^r_X,n) &= \zeta^*(X,n-r). \end{aligned}$$

This suggests that the conjectures $\mathbf{VO}(X,n)$ and $\mathbf{C}(X,n)$ should also satisfy the corresponding compatibilities. We verify in this section that this is indeed the case.

6.1. Lemma. *Let $n<0$.*

1) If $X = \coprod_{1\le i\le r} X_i$ is a finite disjoint union of arithmetic schemes, then

$$\mathbf{L}^c(X_{\acute{e}t},n) \iff \mathbf{L}^c(X_{i,\acute{e}t},n) \text{ for all } i.$$

2) For a closed-open decomposition $Z\not\hookrightarrow X\hookleftarrow U$, if two out of three conjectures

$$\mathbf{L}^c(X_{\acute{e}t},n),\quad \mathbf{L}^c(Z_{\acute{e}t},n),\quad \mathbf{L}^c(U_{\acute{e}t},n)$$

hold, then the third holds as well.

3) For an arithmetic scheme X and any $r\ge 0$, one has

$$\mathbf{L}^c(\mathbb{A}^r_{X,\acute{e}t},n) \iff \mathbf{L}^c(X_{\acute{e}t},n-r).$$

Proof. We already verified this in [Bes2020, Lemma 8.7]. ☐

6.2. Lemma.

1) *If* $X = \coprod_{1\le i\le r} X_i$ *is a finite disjoint union of arithmetic schemes, then*

$$Reg_{X,n} = \bigoplus_{1\le i\le r} Reg_{X_i,n} \colon \bigoplus_{1\le i\le r} R\Gamma(X_{i,\acute{e}t}, \mathbb{R}^c(n)) \to \bigoplus_{i\le i\le r} R\Gamma_{BM}(G_{\mathbb{R}}, X_i(\mathbb{C}), \mathbb{R}(n))[1].$$

In particular,

$$\mathbf{B}(X,n) \iff \mathbf{B}(X_i,n) \text{ for all } i.$$

2) *For a closed-open decomposition of arithmetic schemes* $Z \not\hookrightarrow X \hookleftarrow U$*, the corresponding regulators give a morphism of distinguished triangles*

$$\begin{tikzcd}
R\Gamma(Z_{\acute{e}t}, \mathbb{R}^c(n)) \arrow[r] \arrow[d, "Reg_{Z,n}"] & R\Gamma(X_{\acute{e}t}, \mathbb{R}^c(n)) \arrow[r] \arrow[d, "Reg_{X,n}"] & R\Gamma(U_{\acute{e}t}, \mathbb{R}^c(n)) \arrow[r] \arrow[d, "Reg_{U,n}"] & \cdots[1] \arrow[d, "Reg_{Z,n}[1]"] \\
R\Gamma_{BM}(G_{\mathbb{R}}, Z(\mathbb{C}), \mathbb{R}(n))[1] \arrow[r] & R\Gamma_{BM}(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[1] \arrow[r] & R\Gamma_{BM}(G_{\mathbb{R}}, U(\mathbb{C}), \mathbb{R}(n))[1] \arrow[r] & \cdots[2]
\end{tikzcd}$$

In particular, if two out of three conjectures

$$\mathbf{B}(X,n), \quad \mathbf{B}(Z,n), \quad \mathbf{B}(U,n)$$

hold, then the third holds as well.

3) *For any* $r \ge 0$*, the diagram*

$$\begin{tikzcd}
R\Gamma(X_{\acute{e}t}, \mathbb{R}^c(n-r))[2r] \arrow[r, "\cong"] \arrow[d, "Reg_{X,n-r}"] & R\Gamma(\mathbb{A}^r_{X,\acute{e}t}, \mathbb{R}^c(n)) \arrow[d, "Reg_{\mathbb{A}^r_X,n}"] \\
R\Gamma_{BM}(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n-r))[2r] \arrow[r, "\cong"] & R\Gamma_{BM}(G_{\mathbb{R}}, \mathbb{A}^r_X(\mathbb{C}), \mathbb{R}(n))
\end{tikzcd}$$

commutes. In particular, one has

$$\mathbf{B}(\mathbb{A}^r_X, n) \iff \mathbf{B}(X, n-r).$$

Proof. Part 1) is clear, since all cohomologies involved in the definition of $Reg_{X,n}$ decompose into direct sums over $i = 1,\ldots r$. Parts 2) and 3) boil down to the corresponding functoriality properties for the KLM morphism (2.1), namely that it commutes with proper pushforwards and flat pullbacks. For this we refer to [Wei2017, Lemma 3, Lemma 4], and it may be also verified directly from the KLM formula. For closed-open decompositions, the distinguished triangle

$$R\Gamma(Z_{\acute{e}t}, \mathbb{R}^c(n)) \to R\Gamma(X_{\acute{e}t}, \mathbb{R}^c(n)) \to R\Gamma(U_{\acute{e}t}, \mathbb{R}^c(n)) \to R\Gamma(Z_{\acute{e}t}, \mathbb{R}^c(n))[1]$$

comes precisely from proper pushforward along $Z \hookrightarrow X$ and flat pullback along $U \hookrightarrow X$ (see [Gei2010b, Corollary 7.2] and [Blo1986a, §3]). Similarly, the quasi-isomorphism $R\Gamma(X_{\acute{e}t}, \mathbb{R}^c(n-r))[2r] \cong R\Gamma(\mathbb{A}^r_{X,\acute{e}t}, \mathbb{R}^c(n))$ comes from the flat pullback along $p\colon \mathbb{A}^r_X \to X$. ☐

6.3. Proposition. *For each arithmetic scheme* X *below and* $n < 0$*, assume* $\mathbf{L}^c(X_{\acute{e}t}, n)$*,* $\mathbf{B}(X,n)$*, and the meromorphic continuation of* $\zeta(X,s)$ *around* $s = n$*.*

1) *If* $X = \coprod_{1\le i\le r} X_i$ *is a finite disjoint union of arithmetic schemes, then*

$$\mathbf{VO}(X,n) \iff \mathbf{VO}(X_i,n) \text{ for all } i.$$

2) For a closed-open decomposition $Z \not\hookrightarrow X \hookleftarrow U$, if two out of three conjectures

$$\mathbf{VO}(X,n), \quad \mathbf{VO}(Z,n), \quad \mathbf{VO}(U,n)$$

hold, then the third holds as well.

3) For any $r \geq 0$, one has

$$\mathbf{VO}(\mathbb{A}^r_X, n) \iff \mathbf{VO}(X, n-r).$$

Proof. We already observed in proposition 3.4 above that under the conjecture $\mathbf{B}(X,n)$ we can rewrite $\mathbf{VO}(X,n)$ as

$$\operatorname{ord}_{s=n} \zeta(X,s) = \chi(R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))).$$

In part 1), we have

$$\operatorname{ord}_{s=n} \zeta(X,s) = \sum_{1\leq i\leq r} \operatorname{ord}_{s=n} \zeta(X_i,s),$$

and for the corresponding $G_{\mathbb{R}}$-equivariant cohomology,

$$R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)) = \bigoplus_{1\leq i\leq r} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)).$$

The statement follows from the additivity of Euler characteristic:

$$\begin{array}{ccc}
\operatorname{ord}_{s=n} \zeta(X,s) & \xlongequal{\mathbf{VO}(X,n)} & \chi(R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))) \\
\| & & \| \\
\sum_{1\leq i\leq r} \operatorname{ord}_{s=n} \zeta(X_i,s) & \xlongequal{\forall i \mathbf{VO}(X_i,n)} & \sum_{1\leq i\leq r} \chi(R\Gamma_c(G_{\mathbb{R}}, X_i(\mathbb{C}), \mathbb{R}(n)))
\end{array}$$

Similarly in part 2), we may consider the distinguished triangle

$$R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), \mathbb{R}(n)) \to R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)) \to R\Gamma_c(G_{\mathbb{R}}, Z(\mathbb{C}), \mathbb{R}(n)) \to R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), \mathbb{R}(n))[1]$$

and the additivity of Euler characteristic gives

$$\begin{array}{ccc}
\operatorname{ord}_{s=n} \zeta(X,s) & \xlongequal{\mathbf{VO}(X,n)} & \chi(R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))) \\
\| & & \| \\
\operatorname{ord}_{s=n} \zeta(Z,s) & \xlongequal{\mathbf{VO}(Z,n)} & \chi(R\Gamma_c(G_{\mathbb{R}}, Z(\mathbb{C}), \mathbb{R}(n))) \\
+ & & + \\
\operatorname{ord}_{s=n} \zeta(U,s) & \xlongequal{\mathbf{VO}(Z,n)} & \chi(R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), \mathbb{R}(n)))
\end{array}$$

Finally, in part 3), assume for simplicity that $X_{\mathbb{C}}$ is connected of dimension $d_{\mathbb{C}}$. Then the Poincaré duality and homotopy invariance of the usual cohomology without compact support give us

$$\begin{aligned}
R\Gamma_c(G_{\mathbb{R}}, \mathbb{A}^r(\mathbb{C}) \times X(\mathbb{C}), \mathbb{R}(n)) &\overset{\text{P.D.}}{\cong} R\mathrm{Hom}(R\Gamma(G_{\mathbb{R}}, \mathbb{A}^r(\mathbb{C}) \times X(\mathbb{C}), \mathbb{R}(d_{\mathbb{C}}+r-n)), \mathbb{R})[-2d_{\mathbb{C}}-2r] \overset{\text{H.I.}}{\cong} \\
& R\mathrm{Hom}(R\Gamma(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(d_{\mathbb{C}}+r-n)), \mathbb{R})[-2d_{\mathbb{C}}-2r] \overset{\text{P.D.}}{\cong} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n-r))[-2r].
\end{aligned}$$

The twist $[-2r]$ is even, hence it does not affect the Euler characteristic, so that we obtain

$$\begin{array}{ccc}
\operatorname{ord}_{s=n} \zeta(\mathbb{A}^r_X,s) & \xlongequal{\mathbf{VO}(\mathbb{A}^r_X,n)} & \chi(R\Gamma_c(G_{\mathbb{R}}, \mathbb{A}^r(\mathbb{C}) \times X(\mathbb{C}), \mathbb{R}(n))) \\
\| & & \| \\
\operatorname{ord}_{s=n-r} \zeta(X,s) & \xlongequal{\mathbf{VO}(X,n-r)} & \chi(R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n-r)))
\end{array}$$

$\square$

6.4. Remark. Recall that the formula that appears in the original statement of $\mathbf{VO}(X,n)$ reads

$$\operatorname{ord}_{s=n}\zeta(X,s) = \chi'(R\Gamma_{W,c}(X,\mathbb{Z}(n))) := \sum_{i\in\mathbb{Z}}(-1)^i\cdot i\cdot \operatorname{rk}_{\mathbb{Z}} H^i_{W,c}(X,\mathbb{Z}(n)). \tag{6.4}$$

The conjecture $\mathbf{B}(X,n)$ in the above argument is needed to rewrite this in terms of the usual Euler characteristic. We used $\chi(R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{R}(n)))$, but we could do the same with $\chi(R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R})[1])$.

The least interesting part 1) of the previous proposition could be proved directly from (6.4), since $H^i_{W,c}(X,\mathbb{Z}(n)) = \bigoplus_j H^i_{W,c}(X_j,\mathbb{Z}(n))$. Parts 2) and 3) would be problematic to prove directly from (6.4) without assuming $\mathbf{B}(X,n)$, since the secondary Euler characteristic $\chi'(-)$ does not behave as the usual Euler characteristic $\chi(-)$. In particular, it is not additive for distinguished triangles.

Our next goal is to prove similar compatibilities for the special value conjecture $\mathbf{C}(X,n)$, the same way it was done in proposition 6.3 for $\mathbf{VO}(X,n)$. We will split the proof into three technical lemmas 6.5, 6.6, 6.7, each for the corresponding compatibility. We briefly recall the construction of our Weil-étale complex. It fits in the following diagram in the derived category $\mathbf{D}(\mathbb{Z})$ with distinguished triangles:

$$\begin{tikzcd}
 & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]) \arrow[r]\arrow[d, "\alpha_{X,n}"] & 0 \arrow[d] & \\
 & R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n)) \arrow[r, "u_\infty^*"]\arrow[d] & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \arrow[d, "id"] & \\
R\Gamma_{W,c}(X,\mathbb{Z}(n)) \arrow[r] & R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[r, dashed, "i_\infty^*"]\arrow[d] & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \arrow[r]\arrow[d] & R\Gamma_{W,c}(X,\mathbb{Z}(n))[1] \\
 & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]) \arrow[r] & 0 &
\end{tikzcd}$$

For further details, the reader may consult [Bes2020].

6.5. Lemma. *Let $n<0$ and let $X=\coprod_{1\le i\le r}X_i$ be a finite disjoint union of arithmetic schemes. Assume $\mathbf{L}^c(X_{\acute{e}t},n)$ and $\mathbf{B}(X,n)$. Then there is a quasi-isomorphism of complexes*

$$\bigoplus_{1\le i\le r} R\Gamma_{W,c}(X_i,\mathbb{Z}(n)) \cong R\Gamma_{W,c}(X,\mathbb{Z}(n)),$$

which after passing to the determinants gives a commutative diagram

$$\begin{tikzcd}
\mathbb{R}\otimes_{\mathbb{R}}\cdots\otimes_{\mathbb{R}}\mathbb{R} \arrow[r, "x_1\otimes\cdots\otimes x_r\mapsto x_1\cdots x_r", "\cong"'] \arrow[d, "\lambda_{X_1,n}\otimes\cdots\otimes\lambda_{X_r,n}", "\cong"'] & \mathbb{R} \arrow[d, "\lambda_{X,n}", "\cong"'] \\
\bigotimes_{1\le i\le r}(\det_{\mathbb{Z}} R\Gamma_{W,c}(X_i,\mathbb{Z}(n)))\otimes_{\mathbb{Z}}\mathbb{R} \arrow[r, "\cong"] & (\det_{\mathbb{Z}} R\Gamma_{W,c}(X_i,\mathbb{Z}(n)))\otimes_{\mathbb{Z}}\mathbb{R}
\end{tikzcd} \tag{6.5}$$

Proof. From the construction of $R\Gamma_{W,c}(X,\mathbb{Z}(n))$ it is clear that for $X=\coprod_{1\le i\le r}X_i$ all involved cohomologies decompose into the corresponding direct sum over $i=1,\dots,r$, and at the end after tensoring with $\mathbb{R}$ one obtains a commutative diagram

$$\begin{tikzcd}
\bigoplus_i \left(\begin{array}{c} R\Gamma_c(G_{\mathbb{R}}, X_i(\mathbb{C}), \mathbb{R}(n))[-2] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X_i(\mathbb{C}), \mathbb{R}(n))[-1] \end{array}\right) \arrow[r, "\cong"] \arrow[d, "\cong"', "\bigoplus_i Reg^\vee_{X_i,n}[-1]\oplus id"] & \begin{array}{c} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-2] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \end{array} \arrow[d, "\cong"', "Reg^\vee_{X,n}[-1]\oplus id"] \\
\bigoplus_i \left(\begin{array}{c} R\mathrm{Hom}(R\Gamma(X_{i,\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X_i(\mathbb{C}), \mathbb{R}(n))[-1] \end{array}\right) \arrow[r, "\cong"] \arrow[d, "\cong"', "\text{split}"] & \begin{array}{c} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \end{array} \arrow[d, "\cong"', "\text{split}"] \\
\bigoplus_i R\Gamma_{W,c}(X_i, \mathbb{R}(n)) \arrow[r, "\cong"] & R\Gamma_{W,c}(X, \mathbb{R}(n))
\end{tikzcd}$$

Taking the determinants, we obtain (6.5). ∎

6.6. Lemma. *Let $n < 0$ and let $Z \not\hookrightarrow X \hookleftarrow U$ be a closed-open decomposition of arithmetic schemes, such that the conjectures*

$$\mathbf{L}^c(U_{\acute{e}t}, n),\ \mathbf{L}^c(X_{\acute{e}t}, n),\ \mathbf{L}^c(Z_{\acute{e}t}, n),$$
$$\mathbf{B}(U, n),\ \mathbf{B}(X, n),\ \mathbf{B}(Z_{\acute{e}t}, n)$$

hold (in each case, it is enough to assume two out of three thanks to lemmas 6.1 and 6.2). Then there is an isomorphism of determinants

$$\det_{\mathbb{Z}} R\Gamma_{W,c}(U, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} R\Gamma_{W,c}(Z, \mathbb{Z}(n)) \cong \det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)) \tag{6.6}$$

making the following diagram commute up to signs:

$$\begin{tikzcd}
\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} \arrow[r, "x\otimes y\mapsto xy"] \arrow[d, "\cong"', "\lambda_{U,n}\otimes\lambda_{Z,n}"] & \mathbb{R} \arrow[d, "\cong"', "\lambda_{X,n}"] \\
\begin{array}{c} (\det_{\mathbb{Z}} R\Gamma_{W,c}(U, \mathbb{Z}(n))) \otimes_{\mathbb{Z}} \mathbb{R} \\ \otimes_{\mathbb{R}} \\ (\det_{\mathbb{Z}} R\Gamma_{W,c}(Z, \mathbb{Z}(n))) \otimes_{\mathbb{Z}} \mathbb{R} \end{array} \arrow[r, "\cong"] & (\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n))) \otimes_{\mathbb{Z}} \mathbb{R}
\end{tikzcd} \tag{6.7}$$

Proof. Morally, we expect in this situation a distinguished triangle of the form

$$R\Gamma_{W,c}(U, \mathbb{Z}(n)) \to R\Gamma_{W,c}(X, \mathbb{Z}(n)) \to R\Gamma_{W,c}(Z, \mathbb{Z}(n)) \to R\Gamma_{W,c}(U, \mathbb{Z}(n))[1]. \tag{6.8}$$

However, even the complex $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ was constructed in [Bes2020] up to a non-canonical isomorphism in the derived category $\mathbf{D}(\mathbb{Z})$, so this is problematic. In the absence of a better definition, we will construct the isomorphism (6.6) in an ad hoc manner.

A closed-open decomposition $Z \not\hookrightarrow X \hookleftarrow U$ gives us distinguished triangles

$$R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n)) \longrightarrow R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)) \longrightarrow R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)) \longrightarrow R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n))[1]$$
$$R\Gamma_c(U_{\acute{e}t}, \mathbb{Z}(n)) \longrightarrow R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \longrightarrow R\Gamma_c(Z_{\acute{e}t}, \mathbb{Z}(n)) \longrightarrow R\Gamma_c(U_{\acute{e}t}, \mathbb{Z}(n))[1]$$
$$R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), \mathbb{R}(n)) \to R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)) \to R\Gamma_c(G_{\mathbb{R}}, Z(\mathbb{C}), \mathbb{R}(n)) \to R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), \mathbb{R}(n))[1]$$

The first triangle is [Gei2010b, Corollary 7.2], and it means that $R\Gamma(-, \mathbb{Z}^c(n))$ behaves like Borel–Moore homology, while the following two triangles are the usual ones for cohomology with compact support. These fit together in a commutative diagram displayed on figure 1 below (page 26). For brevity, we denote $R\mathrm{Hom}(X, Y)$ by $[X, Y]$ in the diagram. Similarly, figure 2 displays the same diagram tensored with $\mathbb{Q}$.

In this diagram, we start from the morphism of triangles $(\alpha_{U,n}, \alpha_{X,n}, \alpha_{Z,n})$, and then take the respective cones $R\Gamma_{fg}(-, \mathbb{Z}(n))$. In fact, by [Bes2020, Proposition 5.6], these cones are well-defined up to a *unique* isomorphism in the derived category $\mathbf{D}(\mathbb{Z})$, and the same argument shows that the induced morphisms of complexes

$$R\Gamma_{fg}(U, \mathbb{Z}(n)) \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \to R\Gamma_{fg}(Z, \mathbb{Z}(n)) \to R\Gamma_{fg}(U, \mathbb{Z}(n))[1] \tag{6.9}$$

are also uniquely defined (see [Bes2020, Corollary A.3]). A priori, it does not have to be a distinguished triangle*, but we claim that it induces a long exact sequence in cohomology.

For this note that tensoring the diagram with $\mathbb{Z}/m\mathbb{Z}$ gives us an isomorphism

$$\begin{tikzcd}
R\Gamma_c(U_{ét}, \mathbb{Z}/m\mathbb{Z}(n)) \arrow[r] \arrow[d, "\cong"] & R\Gamma_c(X_{ét}, \mathbb{Z}/m\mathbb{Z}(n)) \arrow[r] \arrow[d, "\cong"] & R\Gamma_c(Z_{ét}, \mathbb{Z}/m\mathbb{Z}(n)) \arrow[r] \arrow[d, "\cong"] & R\Gamma_c(U_{ét}, \mathbb{Z}/m\mathbb{Z}(n))[1] \arrow[d, "\cong"] \\
R\Gamma_{fg}(U, \mathbb{Z}(n)) \otimes^{\mathbf{L}}_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \arrow[r] & R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes^{\mathbf{L}}_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \arrow[r] & R\Gamma_{fg}(Z, \mathbb{Z}(n)) \otimes^{\mathbf{L}}_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \arrow[r] & R\Gamma_{fg}(U, \mathbb{Z}(n)) \otimes^{\mathbf{L}}_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}[1]
\end{tikzcd}$$

More generally, for each prime p we may take the corresponding derived p-adic completions (see [BS2015] and [Stacks, Tag 091N])

$$R\Gamma_{fg}(-, \mathbb{Z}(n))^\wedge_p := R \varprojlim_k (R\Gamma_{fg}(-, \mathbb{Z}(n)) \otimes^{\mathbf{L}}_{\mathbb{Z}} \mathbb{Z}/p^k\mathbb{Z}),$$

and these give us a distinguished triangle for each prime p

$$R\Gamma_{fg}(U, \mathbb{Z}(n))^\wedge_p \to R\Gamma_{fg}(X, \mathbb{Z}(n))^\wedge_p \to R\Gamma_{fg}(Z, \mathbb{Z}(n))^\wedge_p \to R\Gamma_{fg}(U, \mathbb{Z}(n))^\wedge_p[1].$$

On the level of cohomology, there are natural isomorphisms [Stacks, Tag 0A06]

$$H^i(R\Gamma_{fg}(-, \mathbb{Z}(n))^\wedge_p) \cong H^i_{fg}(-, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

In particular, for each p there is a long exact sequence of cohomology groups

$$\cdots \to H^i_{fg}(U, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to H^i_{fg}(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to H^i_{fg}(Z, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to H^{i+1}_{fg}(U, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \cdots$$

induced by (6.9). But now since the groups $H^i_{fg}(-, \mathbb{Z}(n))$ are finitely generated, by flatness of $\mathbb{Z}_p$ this implies that the sequence

$$\cdots \to H^i_{fg}(U, \mathbb{Z}(n)) \to H^i_{fg}(X, \mathbb{Z}(n)) \to H^i_{fg}(Z, \mathbb{Z}(n)) \to H^{i+1}_{fg}(U, \mathbb{Z}(n)) \to \cdots \tag{6.10}$$

is exact.

*Taking naively a "cone of a morphism of distinguished triangles"

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & X[1] \arrow[d] \\
X' \arrow[r] \arrow[d, dashed] & Y' \arrow[r] \arrow[d, dashed] & Z' \arrow[r] \arrow[d, dashed] & X'[1] \arrow[d, dashed] \\
X'' \arrow[r, dashed] \arrow[d, dashed] & Y'' \arrow[r, dashed] \arrow[d, dashed] & Z'' \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\text{(ac)}"] & X''[1] \arrow[d, dashed] \\
X[1] \arrow[r] & Y[1] \arrow[r] & Z[1] \arrow[r] & X[2]
\end{tikzcd}$$

normally *does not* give a distinguished triangle $X'' \to Y'' \to Z'' \to X''[1]$, as discussed in [Nee1991]. Here we are dealing with notorious issues associated to working with the classical derived (1-)categories.

Now we consider the diagram

$$\begin{array}{ccccccc}
\tau_{\leq m}R\Gamma_c(G_{\mathbb{R}},U(\mathbb{C}),\mathbb{Z}(n))[-1] & \longrightarrow & R\Gamma_{W,c}(U,\mathbb{Z}(n)) & \longrightarrow & \tau_{\leq m}R\Gamma_{fg}(U,\mathbb{Z}(n)) & \longrightarrow & \tau_{\leq m}R\Gamma_c(G_{\mathbb{R}},U(\mathbb{C}),\mathbb{Z}(n)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tau_{\leq m}R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n))[-1] & \longrightarrow & R\Gamma_{W,c}(X,\mathbb{Z}(n)) & \longrightarrow & \tau_{\leq m}R\Gamma_{fg}(X,\mathbb{Z}(n)) & \longrightarrow & \tau_{\leq m}R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tau_{\leq m}R\Gamma_c(G_{\mathbb{R}},Z(\mathbb{C}),\mathbb{Z}(n))[-1] & \longrightarrow & R\Gamma_{W,c}(Z,\mathbb{Z}(n)) & \longrightarrow & \tau_{\leq m}R\Gamma_{fg}(Z,\mathbb{Z}(n)) & \longrightarrow & \tau_{\leq m}R\Gamma_c(G_{\mathbb{R}},Z(\mathbb{C}),\mathbb{Z}(n)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tau_{\leq m}R\Gamma_c(G_{\mathbb{R}},U(\mathbb{C}),\mathbb{Z}(n)) & \longrightarrow & R\Gamma_{W,c}(U,\mathbb{Z}(n))[1] & \rightarrow & \tau_{\leq m}R\Gamma_{fg}(U,\mathbb{Z}(n))[1] & \rightarrow & \tau_{\leq m}R\Gamma_c(G_{\mathbb{R}},U(\mathbb{C}),\mathbb{Z}(n))[1]
\end{array}$$

Here we took truncations for m big enough similarly to the proof of lemma 4.3. There are canonical isomorphisms

$$\begin{aligned}
\det_{\mathbb{Z}} R\Gamma_{W,c}(U,\mathbb{Z}(n)) &\cong \det_{\mathbb{Z}}(\tau_{\leq m}R\Gamma_c(G_{\mathbb{R}},U(\mathbb{C}),\mathbb{Z}(n))[-1]) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}}(\tau_{\leq m}R\Gamma_{fg}(U,\mathbb{Z}(n))),\\
\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)) &\cong \det_{\mathbb{Z}}(\tau_{\leq m}R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n))[-1]) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}}(\tau_{\leq m}R\Gamma_{fg}(X,\mathbb{Z}(n))),\\
\det_{\mathbb{Z}} R\Gamma_{W,c}(Z,\mathbb{Z}(n)) &\cong \det_{\mathbb{Z}}(\tau_{\leq m}R\Gamma_c(G_{\mathbb{R}},Z(\mathbb{C}),\mathbb{Z}(n))[-1]) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}}(\tau_{\leq m}R\Gamma_{fg}(Z,\mathbb{Z}(n))),\\
\det_{\mathbb{Z}}(\tau_{\leq m}R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n))) &\cong \det_{\mathbb{Z}}(\tau_{\leq m}R\Gamma_c(G_{\mathbb{R}},U(\mathbb{C}),\mathbb{Z}(n))) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}}(\tau_{\leq m}R\Gamma_c(G_{\mathbb{R}},Z(\mathbb{C}),\mathbb{Z}(n))),\\
\det_{\mathbb{Z}}(\tau_{\leq m}R\Gamma_{fg}(X,\mathbb{Z}(n))) &\cong \det_{\mathbb{Z}}(\tau_{\leq m}R\Gamma_{fg}(U,\mathbb{Z}(n))) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}}(\tau_{\leq m}R\Gamma_{fg}(Z,\mathbb{Z}(n))).
\end{aligned}$$

Here the first four isomorphisms come from true distinguished triangles, while the last isomorphism comes from the cohomology long exact sequence (6.10), which gives an isomorphism

$$\bigotimes_{i\leq m}\Big(\det_{\mathbb{Z}} H^i_{fg}(U,\mathbb{Z}(n))^{(-1)^i} \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} H^i_{fg}(X,\mathbb{Z}(n))^{(-1)^{i+1}} \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} H^i_{fg}(Z,\mathbb{Z}(n))^{(-1)^i}\Big) \cong \mathbb{Z}.$$

We may rearrange the terms *at the cost of introducing a* ± 1 *sign*, to obtain

$$\begin{aligned}
\det_{\mathbb{Z}}(\tau_{\leq m}R\Gamma_{fg}(X,\mathbb{Z}(n))) &\cong \bigotimes_{i\leq m} \det_{\mathbb{Z}} H^i_{fg}(X,\mathbb{Z}(n)) \cong \\
&\bigotimes_{i\leq m} \det_{\mathbb{Z}} H^i_{fg}(U,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \bigotimes_{i\leq m} \det_{\mathbb{Z}} H^i_{fg}(Z,\mathbb{Z}(n)) \cong \\
&\qquad \det_{\mathbb{Z}}(\tau_{\leq m}R\Gamma_{fg}(U,\mathbb{Z}(n))) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}}(\tau_{\leq m}R\Gamma_{fg}(Z,\mathbb{Z}(n))).
\end{aligned}$$

All the above gives us the desired isomorphism of integral determinants (6.6).

Now we consider the following diagram with distinguished rows:

$$\begin{array}{ccccccc}
\begin{matrix} R\Gamma_c(G_{\mathbb{R}},U(\mathbb{C}),\mathbb{R}(n))[-2] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}},U(\mathbb{C}),\mathbb{R}(n))[-1] \end{matrix} & \longrightarrow & \begin{matrix} R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{R}(n))[-2] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{R}(n))[-1] \end{matrix} & \longrightarrow & \begin{matrix} R\Gamma_c(G_{\mathbb{R}},Z(\mathbb{C}),\mathbb{R}(n))[-2] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}},Z(\mathbb{C}),\mathbb{R}(n))[-1] \end{matrix} & \longrightarrow & \cdots \\
\cong\downarrow{\scriptstyle Reg^{\vee}_{U,n}[-1]\oplus id} & & \cong\downarrow{\scriptstyle Reg^{\vee}_{X,n}[-1]\oplus id} & & \cong\downarrow{\scriptstyle Reg^{\vee}_{Z,n}[-1]\oplus id} & & \downarrow \\
\begin{matrix} R\mathrm{Hom}(R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R})[-1] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}},U(\mathbb{C}),\mathbb{R}(n))[-1] \end{matrix} & \rightarrow & \begin{matrix} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R})[-1] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{R}(n))[-1] \end{matrix} & \rightarrow & \begin{matrix} R\mathrm{Hom}(R\Gamma(Z_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R})[-1] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}},Z(\mathbb{C}),\mathbb{R}(n))[-1] \end{matrix} & \rightarrow & \cdots \\
\cong\downarrow{\scriptstyle \text{split}} & & \cong\downarrow{\scriptstyle \text{split}} & & \cong\downarrow{\scriptstyle \text{split}} & & \downarrow \\
R\Gamma_{W,c}(U,\mathbb{R}(n)) & \longrightarrow & R\Gamma_{W,c}(X,\mathbb{R}(n)) & \longrightarrow & R\Gamma_{W,c}(Z,\mathbb{R}(n)) & \longrightarrow & \cdots
\end{array}$$

Here the three squares with regulators involved commute thanks to lemma 6.2. Taking the determinants, we obtain (6.7), using the compatibility of determinants with distinguished triangles. We note that we did not construct an integral distinguished triangle (6.8); instead we only have that the bottom arrow in (6.7) is induced by the ad hoc isomorphism of determinants (6.6). $\square$

6.7. Lemma. *For $n < 0$ and $r \geq 0$, let X be an arithmetic scheme satisfying $\mathbf{L}^c(X_{\acute{e}t}, n-r)$ and $\mathbf{B}(X, n-r)$. Then there is a natural quasi-isomorphism of complexes*

$$R\Gamma_{W,c}(\mathbb{A}^r_X, \mathbb{Z}(n)) \cong R\Gamma_{W,c}(X, \mathbb{Z}(n-r))[-2r], \tag{6.11}$$

which after passing to the determinants makes the following diagram commute:

$$\begin{tikzcd} & \mathbb{R} \arrow[dl, "\lambda_{\mathbb{A}^r_X, n}"', "\cong"] \arrow[dr, "\lambda_{X,n-r}", "\cong"'] & \\ (\det_{\mathbb{Z}} R\Gamma_{W,c}(\mathbb{A}^r_X, \mathbb{Z}(n))) \otimes_{\mathbb{Z}} \mathbb{R} \arrow[rr, "\cong"] & & (\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n-r))) \otimes_{\mathbb{Z}} \mathbb{R} \end{tikzcd} \tag{6.12}$$

Proof. We refer to figure 3 below (page 28) that shows how the flat morphism $p\colon \mathbb{A}^r_X \to X$ induces the desired quasi-isomorphism (6.11). Everything comes down to the homotopy property of motivic cohomology, namely the fact that p induces a quasi-isomorphism

$$p^*\colon R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n-r))[2r] \xrightarrow{\cong} R\Gamma(\mathbb{A}^r_{X,\acute{e}t}, \mathbb{Z}^c(n))$$

—for this see e.g. [Mor2014, Lemma 5.11]. After passing to real coefficients, we obtain the following diagram:

$$\begin{tikzcd} \begin{array}{c} R\Gamma_c(G_{\mathbb{R}}, \mathbb{A}^r_X(\mathbb{C}), \mathbb{R}(n))[-2] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, \mathbb{A}^r_X(\mathbb{C}), \mathbb{R}(n))[-1] \end{array} \arrow[r, "\cong"] \arrow[d, "Reg^\vee_{\mathbb{A}^r_X,n}[-1]\oplus id", "\cong"'] & \begin{array}{c} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n-r))[-2][-2r] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n-r))[-1][-2r] \end{array} \arrow[d, "Reg^\vee_{X,n-r}[-1][-2r]\oplus id", "\cong"'] \\ \begin{array}{c} R\mathrm{Hom}(R\Gamma(\mathbb{A}^r_{X,\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, \mathbb{A}^r_X(\mathbb{C}), \mathbb{R}(n))[-1] \end{array} \arrow[r, "\cong"] \arrow[d, "\text{split}", "\cong"'] & \begin{array}{c} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n-r))[2r], \mathbb{R})[-1] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n-r))[-1][-2r] \end{array} \arrow[d, "\text{split}", "\cong"'] \\ R\Gamma_{W,c}(\mathbb{A}^r_X, \mathbb{R}(n)) \arrow[r, "\cong"] & R\Gamma_{W,c}(X, \mathbb{R}(n-r))[-2r] \end{tikzcd}$$

Here the first square commutes by the compatibility of the regulator with affine bundles (lemma 6.2), and the second square commutes because the quasi-isomorphism (6.11) gives compatible splittings (again, see figure 3 below). Taking the determinants, we obtain the desired commutative diagram (6.12). $\square$

6.8. Theorem. *For an arithmetic scheme X and $n < 0$, assume $\mathbf{L}^c(X_{\acute{e}t}, n)$, $\mathbf{B}(X, n)$, and the meromorphic continuation of $\zeta(X, s)$ around $s = n$.*

1) If $X = \coprod_{1 \leq i \leq r} X_i$ is a finite disjoint union of arithmetic schemes, then

$$\mathbf{C}(X, n) \iff \mathbf{C}(X_i, n) \text{ for all } i.$$

2) For a closed-open decomposition $Z \not\hookrightarrow X \hookleftarrow U$, if two out of three conjectures

$$\mathbf{C}(X, n), \quad \mathbf{C}(Z, n), \quad \mathbf{C}(U, n)$$

hold, then the third holds as well.

3) For any $r \geq 0$, one has

$$\mathbf{C}(\mathbb{A}^r_X, n) \iff \mathbf{C}(X, n-r).$$

Proof. Follows from the previous lemmas 6.5, 6.6, 6.7, together with the respective identities for zeta functions (6.1), (6.2), (6.3). ☐

6.9. Remark. As a formal consequence of compatibility with closed-open decompositions, if we apply it to the canonical closed embedding $X_{red} \hookrightarrow X$, then we conclude that $R\Gamma_{W,c}(X, \mathbb{Z}(n)) \cong R\Gamma_{W,c}(X_{red}, \mathbb{Z}(n))$. This is not surprising, because Weil-étale complexes are constructed from a variant of cycle complexes / higher Chow groups, and these do not distinguish X from X_{red} (unlike, for instance, algebraic K-groups).

This is actually the desired behavior for us, since neither the zeta function does: $\zeta(X, s) = \zeta(X_{red}, s)$.

6.10. Remark. If $X/\mathbb{F}_q$ is a variety over a finite field, then the proof of theorem 6.8 simplifies drastically: we can work with the formula (5.1), and the following properties of motivic cohomology:

1) $R\Gamma(\coprod_i X_{i,\acute{e}t}, \mathbb{Z}^c(n)) \cong \bigoplus_i R\Gamma(X_{i,\acute{e}t}, \mathbb{Z}^c(n))$;

2) triangles $R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n)) \to R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)) \to R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)) \to R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n))[1]$ associated to closed-open decompositions;

3) homotopy invariance $R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n-r))[2r] \cong R\Gamma(\mathbb{A}^r_{X,\acute{e}t}, \mathbb{Z}^c(n))$.

There are no regulators involved in this case, so we do not need the technical lemmas 6.5, 6.6, 6.7.

Considering the projective space $\mathbb{P}^r_X = \mathbb{P}^r_\mathbb{Z} \times X$, we have a formula for the zeta function

$$\zeta(\mathbb{P}^r_X, s) = \prod_{0 \leq i \leq r} \zeta(X, s-i). \tag{6.13}$$

6.11. Corollary (projective bundles). *Let X be an arithmetic scheme, $n < 0$, and $r \geq 0$. For $i = 0, \ldots, r$ assume the conjectures $\mathbf{L}^c(X_{\acute{e}t}, n-i)$, $\mathbf{B}(X, n-i)$, and meromorphic continuation of $\zeta(X, s)$ around $s = n-i$. Then*

$$\mathbf{C}(X, n-i) \text{ for } i = 0, \ldots, r \Longrightarrow \mathbf{C}(\mathbb{P}^r_X, n).$$

Proof. Applied to the closed-open decomposition $\mathbb{P}^{r-1}_X \not\hookrightarrow \mathbb{P}^r_X \hookleftarrow \mathbb{A}^r_X$, theorem 6.8 gives

$$\mathbf{C}(X, n-r) \text{ and } \mathbf{C}(\mathbb{P}^{r-1}_X, n) \Longrightarrow \mathbf{C}(\mathbb{A}^r_X, n) \text{ and } \mathbf{C}(\mathbb{P}^{r-1}_X, n) \Longrightarrow \mathbf{C}(\mathbb{P}^r_X, n).$$

The claim follows by induction on r. (Note that the same inductive argument proves the formula (6.13) from (6.3).) ☐

$$
\begin{tikzcd}[column sep=small]
 & & & & R\Gamma_{W,c}(U,\mathbb{Z}(n)) \arrow[dl] & \\
{[R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]]} \arrow[r, "\alpha_{U,n}"] \arrow[dd] & R\Gamma_c(U_{\acute{e}t},\mathbb{Z}(n)) \arrow[rr] \arrow[dd] \arrow[dr, "u_\infty"] & & R\Gamma_{fg}(U,\mathbb{Z}(n)) \arrow[rr] \arrow[dd, dashed, "\exists!"] \arrow[dl, "i_\infty"'] & & {[R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]]} \arrow[dd] \\
 & & R\Gamma_c(G_{\mathbb{R}},U(\mathbb{C}),\mathbb{Z}(n)) \arrow[dd] & & R\Gamma_{W,c}(X,\mathbb{Z}(n)) \arrow[dl] & \\
{[R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]]} \arrow[r, "\alpha_{X,n}"] \arrow[dd] & R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n)) \arrow[rr] \arrow[dd] \arrow[dr, "u_\infty"] & & R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[rr] \arrow[dd, dashed, "\exists!"] \arrow[dl, "i_\infty"'] & & {[R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]]} \arrow[dd] \\
 & & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \arrow[dd] & & R\Gamma_{W,c}(Z,\mathbb{Z}(n)) \arrow[dl] & \\
{[R\Gamma(Z_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]]} \arrow[r, "\alpha_{Z,n}"] \arrow[dd] & R\Gamma_c(Z_{\acute{e}t},\mathbb{Z}(n)) \arrow[rr] \arrow[dd] \arrow[dr, "u_\infty"] & & R\Gamma_{fg}(Z,\mathbb{Z}(n)) \arrow[rr] \arrow[dd, dashed, "\exists!"] \arrow[dl, "i_\infty"'] & & {[R\Gamma(Z_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]]} \arrow[dd] \\
 & & R\Gamma_c(G_{\mathbb{R}},Z(\mathbb{C}),\mathbb{Z}(n)) \arrow[dd] & & R\Gamma_{W,c}(U,\mathbb{Z}(n))[1] \arrow[dl] & \\
{[R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]]} \arrow[r, "{\alpha_{U,n}[1]}"] & R\Gamma_c(U_{\acute{e}t},\mathbb{Z}(n))[1] \arrow[rr] \arrow[dr, "u_\infty"] & & R\Gamma_{fg}(U,\mathbb{Z}(n))[1] \arrow[rr] \arrow[dl, "{i_\infty[1]}"'] & & {[R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}]} \\
 & & R\Gamma_c(G_{\mathbb{R}},U(\mathbb{C}),\mathbb{Z}(n))[1] & & &
\end{tikzcd}
$$

Figure 1: Diagram induced by a closed-open decomposition $Z \not\hookrightarrow X \hookleftarrow U$

$$
\begin{tikzcd}
 & & & & R\Gamma_{W,c}(U, \mathbb{Q}(n)) \arrow[dl] \arrow[dd] & \\
{[R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]]} \arrow[r] \arrow[dd] & 0 \arrow[rr] \arrow[dr] \arrow[dd] & & R\Gamma_{fg}(U, \mathbb{Q}(n)) \arrow[rr, "\cong"] \arrow[dl, "0"'] \arrow[dd, dashed, "\exists!"] & & {[R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]]} \arrow[dd] \\
 & & R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), \mathbb{Q}(n)) \arrow[dd] & & R\Gamma_{W,c}(X, \mathbb{Q}(n)) \arrow[dl] \arrow[dd] & \\
{[R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]]} \arrow[r] \arrow[dd] & 0 \arrow[rr] \arrow[dr] \arrow[dd] & & R\Gamma_{fg}(X, \mathbb{Q}(n)) \arrow[rr, "\cong"] \arrow[dl, "0"'] \arrow[dd, dashed, "\exists!"] & & {[R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]]} \arrow[dd] \\
 & & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}(n)) \arrow[dd] & & R\Gamma_{W,c}(Z, \mathbb{Q}(n)) \arrow[dl] \arrow[dd] & \\
{[R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]]} \arrow[r] \arrow[dd] & 0 \arrow[rr] \arrow[dr] \arrow[dd] & & R\Gamma_{fg}(Z, \mathbb{Q}(n)) \arrow[rr, "\cong"] \arrow[dl, "0"'] \arrow[dd, dashed, "\exists!"] & & {[R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]]} \arrow[dd] \\
 & & R\Gamma_c(G_{\mathbb{R}}, Z(\mathbb{C}), \mathbb{Q}(n)) \arrow[dd] & & R\Gamma_{W,c}(U, \mathbb{Q}(n))[1] \arrow[dl] & \\
{[R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]]} \arrow[r] & 0 \arrow[rr] \arrow[dr] & & R\Gamma_{fg}(U, \mathbb{Q}(n))[1] \arrow[rr, "\cong"] \arrow[dl, "0"'] & & {[R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}]} \\
 & & R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), \mathbb{Q}(n))[1] & & &
\end{tikzcd}
$$

Figure 2: Diagram induced by a closed-open decomposition $Z \not\hookrightarrow X \hookleftarrow U$, tensored with $\mathbb{Q}$

$$\begin{tikzcd}
 & & & & R\Gamma_{W,c}(\mathbb{A}^r_X,\mathbb{Z}(n)) \arrow[dl] \arrow[dd, dashed, "\cong"] & \\
{[R\Gamma(\mathbb{A}^r_{X,\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]]} \arrow[r, "\alpha_{\mathbb{A}^r_X,n}"] \arrow[dd, "\cong"', "(p^*)^\vee"] & R\Gamma_c(\mathbb{A}^r_{X,\acute{e}t},\mathbb{Z}(n)) \arrow[rr] \arrow[dd, "\cong"', "p_*"] \arrow[dr, "u_\infty"] & & R\Gamma_{fg}(\mathbb{A}^r_X,\mathbb{Z}(n)) \arrow[rr] \arrow[dl, "i_\infty"'] \arrow[dd, dashed, "\cong"] & & {[+1]} \arrow[dd, "\cong"] \\
 & & R\Gamma_c(G_\mathbb{R},\mathbb{A}^r_X(\mathbb{C}),\mathbb{Z}(n)) \arrow[dd, "\cong"', "p_*"] & & R\Gamma_{W,c}(X,\mathbb{Z}(n-r))[-2r] \arrow[dl] & \\
{[R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n-r))[2r],\mathbb{Q}[-2]]} \arrow[r, "{\alpha_{X,n-r}[-2r]}"] & R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n-r))[-2r] \arrow[rr] \arrow[dr, "u_\infty"] & & R\Gamma_{fg}(X,\mathbb{Z}(n-r))[-2r] \arrow[rr] \arrow[dl, "i_\infty"'] & & {[+1]} \\
 & & R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{Z}(n-r))[-2r] & & &
\end{tikzcd}$$

$$\otimes_\mathbb{Z}\mathbb{Q} =$$

$$\begin{tikzcd}
 & & & & R\Gamma_{W,c}(\mathbb{A}^r_X,\mathbb{Q}(n)) \arrow[dl] \arrow[dd, dashed, "\cong"] & \\
{[R\Gamma(\mathbb{A}^r_{X,\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]]} \arrow[r] \arrow[dd, "\cong"', "(p^*)^\vee"] & 0 \arrow[rr] \arrow[dd] \arrow[dr] & & R\Gamma_{fg}(\mathbb{A}^r_X,\mathbb{Q}(n)) \arrow[rr] \arrow[dl, "0"'] \arrow[dd, dashed, "\cong"] & & {[+1]} \arrow[dd, "\cong"] \\
 & & R\Gamma_c(G_\mathbb{R},\mathbb{A}^r_X(\mathbb{C}),\mathbb{Q}(n)) \arrow[dd, "\cong"', "p_*"] & & R\Gamma_{W,c}(X,\mathbb{Q}(n-r))[-2r] \arrow[dl] & \\
{[R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n-r))[2r],\mathbb{Q}[-2]]} \arrow[r] & 0 \arrow[rr] \arrow[dr] & & R\Gamma_{fg}(X,\mathbb{Q}(n-r))[-2r] \arrow[rr] \arrow[dl, "0"'] & & {[+1]} \\
 & & R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{Q}(n-r))[-2r] & & &
\end{tikzcd}$$

Figure 3: Isomorphism $R\Gamma_{W,c}(\mathbb{A}^r_X,\mathbb{Z}(n)) \cong R\Gamma_{W,c}(X,\mathbb{Z}(n-r))[-2r]$ and its splitting after tensoring with $\mathbb{Q}$

7 Unconditional results

Now we apply theorem 6.8 from the previous section in order to prove the main theorem stated in the introduction: the validity of $\mathbf{VO}(X,n)$ and $\mathbf{C}(X,n)$ for all $n<0$ for cellular schemes over certain 1-dimensional bases. In fact, we will construct an even bigger class of schemes $\mathcal{C}(\mathbb{Z})$ whose elements satisfy the conjectures. This approach is motivated by [Mor2014, §5].

7.1. Definition. Let $\mathcal{C}(\mathbb{Z})$ be the full subcategory of the category of arithmetic schemes generated by the following objects:

- the empty scheme $\varnothing$,
- $\operatorname{Spec}\mathbb{F}_q$ for a finite field,
- $\operatorname{Spec}\mathcal{O}_F$ for an abelian number field $F/\mathbb{Q}$,
- curves over finite fields $C/\mathbb{F}_q$,

and the following operations.

$\mathcal{C}0$) X lies in $\mathcal{C}(\mathbb{Z})$ if and only if X_{red} lies in $\mathcal{C}(\mathbb{Z})$.

$\mathcal{C}1$) A finite disjoint union $\coprod_{1\le i\le r} X_i$ lies in $\mathcal{C}(\mathbb{Z})$ if and only if each X_i lies in $\mathcal{C}(\mathbb{Z})$.

$\mathcal{C}2$) Let $Z \not\hookrightarrow X \hookleftarrow U$ be a closed-open decomposition such that $Z_{red,\mathbb{C}}$, $X_{red,\mathbb{C}}$, $U_{red,\mathbb{C}}$ are smooth and quasi-projective. Then if two out of three schemes Z,X,U lie in $\mathcal{C}(\mathbb{Z})$, then the third lies as well.

$\mathcal{C}3$) If X lies in $\mathcal{C}(\mathbb{Z})$, then the affine space $\mathbb{A}^r_X$ also lies in $\mathcal{C}(\mathbb{Z})$ for each $r\ge 0$.

We recall that the condition that $X_{red,\mathbb{C}}$ is smooth and quasi-projective is needed to ensure that the regulator morphism exists (see remark 2.7).

7.2. Proposition. *The conjectures $\mathbf{VO}(X,n)$ and $\mathbf{C}(X,n)$ hold for any $X\in\mathcal{C}(\mathbb{Z})$ and $n<0$.*

Proof. Finite fields satisfy $\mathbf{C}(X,n)$ by example 5.5.

If $X=\operatorname{Spec}\mathcal{O}_F$ for an abelian number field $F/\mathbb{Q}$, then the conjecture $\mathbf{C}(X,n)$ is equivalent to the conjecture of Flach and Morin [FM2018, Conjecture 5.12], which holds unconditionally in this particular case, via reduction to the Tamagawa number conjecture; see [FM2018, §5.8.3], in particular [ibid., Proposition 5.35]. The condition $\mathbf{VO}(X,n)$ is also true in this case (see example 3.7).

If $X=C/\mathbb{F}_q$ is a curve over a finite field, then $\mathbf{C}(X,n)$ holds thanks to theorem 5.3. The conjecture $\mathbf{L}^c(X_{\acute{e}t},n)$ is well-known for curves and essentially goes back to Soulé; see for instance [Gei2017, Proposition 4.3].

Then the fact that the conjectures $\mathbf{L}^c(X_{\acute{e}t},n)$, $\mathbf{B}(X,n)$, $\mathbf{VO}(X,n)$, $\mathbf{C}(X,n)$ are closed under the operations $\mathcal{C}1$)–$\mathcal{C}3$) is lemma 6.1, lemma 6.2, proposition 6.3, and theorem 6.8 respectively. ☐

7.3. Lemma. *Any 0-dimensional arithmetic scheme X lies in $\mathcal{C}(\mathbb{Z})$.*

Proof. Since X is a noetherian scheme of dimension 0, it is a finite disjoint union of $\operatorname{Spec}A_i$ for some artinian local rings A_i. Thanks to $\mathcal{C}1$), we may assume that $X=\operatorname{Spec}A$, and thanks to $\mathcal{C}0$), we may assume that X is reduced. But then $A=k$ is a field. Since X is a scheme of finite type over $\operatorname{Spec}\mathbb{Z}$, we conclude that $X=\operatorname{Spec}\mathbb{F}_q\in\mathcal{C}(\mathbb{Z})$. ☐

7.4. Proposition. *Let B be a 1-dimensional arithmetic scheme. Assume that each of the generic points $\eta\in B$ satisfies one of the following properties:*

a) $\operatorname{char}\kappa(\eta)=p>0$*;*

b) $\operatorname{char}\kappa(\eta)=0$*, and $\kappa(\eta)/\mathbb{Q}$ is an abelian number field.*

Then $B \in \mathcal{C}(\mathbb{Z})$.

Proof. We will see that such a scheme can be obtained from $\operatorname{Spec} \mathcal{O}_F$ for an abelian number field $F/\mathbb{Q}$, or a curve over a finite field $C/\mathbb{F}_q$, using the operations $\mathcal{C}0$), $\mathcal{C}1$), $\mathcal{C}2$) that appear in the definition of $\mathcal{C}(\mathbb{Z})$.

Thanks to $\mathcal{C}0$), we may assume that B is reduced. Consider the normalization $\nu\colon B' \to B$. This is a birational morphism, and there exist open dense subsets $U' \subseteq B'$ and $U \subseteq B$ such that $\nu|_{U'} : U' \xrightarrow{\cong} U$. Now $B \setminus U$ is 0-dimensional, and therefore $B \setminus U \in \mathcal{C}(\mathbb{Z})$ by the previous lemma. Thanks to $\mathcal{C}2$), it is enough to check that $U' \in \mathcal{C}(\mathbb{Z})$, and this would imply $B \in \mathcal{C}(\mathbb{Z})$.

Now U' is a finite disjoint union of normal integral schemes, so by $\mathcal{C}1$) we may assume that U' is integral. Consider the generic point $\eta \in U'$ and the residue field $F = \kappa(\eta)$. There are two distinct cases to consider.

a) If $\operatorname{char} F = p > 0$, then U' is a curve over a finite field, so it lies in $\mathcal{C}(\mathbb{Z})$ by the definition.

b) If $\operatorname{char} F = 0$, then by our assumptions, $F/\mathbb{Q}$ is an abelian number field.

   We note that if $V' \subseteq U'$ is an affine open neighborhood of η, then $U' \setminus V' \in \mathcal{C}(\mathbb{Z})$ by the previous lemma. Therefore, we can assume without loss of generality that U' is affine.

   We have $U' = \operatorname{Spec} \mathcal{O}$, where $\mathcal{O}$ is a finitely generated integrally closed domain. All this means that $\mathcal{O}_F \subseteq \mathcal{O} = \mathcal{O}_{F,S}$ for some finite set S. Now $U' = \operatorname{Spec} \mathcal{O}_F \setminus S$, and $S \in \mathcal{C}(\mathbb{Z})$, so again, everything reduces to the case of $U' = \operatorname{Spec} \mathcal{O}_F$, which lies in $\mathcal{C}(\mathbb{Z})$ by the definition. ☐

7.5. Remark. Schemes as above were considered by Jordan and Poonen in [JP2020], where the authors write down a special value formula at $s = 1$, generalizing the classical class number formula. Namely, they consider the case of B reduced and affine, albeit without requiring $\kappa(\eta)/\mathbb{Q}$ to be abelian.

7.6. Example. If $B = \operatorname{Spec} \mathcal{O}$ for a nonmaximal order $\mathcal{O} \subset \mathcal{O}_F$, where $F/\mathbb{Q}$ is an abelian number field, then our formalism gives a cohomological interpretation of the special values of $\zeta_{\mathcal{O}}(s)$ at $s = n < 0$. This already seems to be a new result.

7.7. Definition. Let $X \to B$ be a B-scheme. We say that X is **B-cellular** if it admits a filtration by closed subschemes

$$X = Z_N \supseteq Z_{N-1} \supseteq \cdots \supseteq Z_0 \supseteq Z_{-1} = \varnothing \tag{7.1}$$

such that $Z_i \setminus Z_{i-1} \cong \coprod_j \mathbb{A}_B^{r_{ij}}$ is a finite union of affine B-spaces.

For instance, projective spaces $\mathbb{P}^r_B$ and in general Grassmannians $\operatorname{Gr}(k, \ell)_B$ are cellular. Many interesting examples of cellular schemes as above arise from actions of algebraic groups on varieties and Białynicki-Birula theorem; for this see [Wen2010] and [Bro2005].

7.8. Proposition. *Let X be a B-cellular arithmetic scheme, where $B \in \mathcal{C}(\mathbb{Z})$, and $X_{red,\mathbb{C}}$ is smooth and quasi-projective. Then $X \in \mathcal{C}(\mathbb{Z})$.*

Proof. Looking at the corresponding cellular decomposition (7.1), we better pass to open complements $U_i = X \setminus Z_i$, to obtain a filtration

$$X = U_{-1} \supseteq U_1 \supseteq \cdots \supseteq U_{N-1} \supseteq U_N = \varnothing,$$

with $U_{i,\mathbb{C}}$ smooth and quasi-projective, being *open* subvarieties in $X_{\mathbb{C}}$. Now we have closed-open decompositions $\coprod_j \mathbb{A}_B^{r_{i,j}} \not\hookrightarrow U_i \hookleftarrow U_{i+1}$, and the claim follows by induction on the length of the cellular decomposition, using operations $\mathcal{C}1$)–$\mathcal{C}3$). ☐

As a corollary of the above, we obtain the following result, stated in the introduction.

7.9. Theorem. *Let B be a 1-dimensional arithmetic scheme satisfying the assumptions of proposition 7.4. If X is a B-cellular arithmetic scheme with smooth and quasi-projective fiber $X_{red,\mathbb{C}}$, then the conjectures* $\mathbf{VO}(X, n)$ *and* $\mathbf{C}(X, n)$ *hold unconditionally for any $n < 0$.*

Proof. Follows from propositions 7.2, 7.4, 7.8. ☐

A Determinants of complexes

In this appendix we include a brief overview of determinants of complexes. The original construction is due to Knudsen and Mumford [KM1976], and other useful expositions may be found in [GKZ1994, Appendix A] and [Kat1993, §2.1].

For our purposes, let R be a commutative ring, which is an integral domain (we will be interested in $R = \mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$). Denote by $\mathcal{P}_{is}(R)$* the category of graded invertible R-modules. It has as its objects pairs (L, r), where L is an invertible R-module (= projective of rank 1) and $r \in \mathbb{Z}$. The morphisms in this category are given by

$$\operatorname{Hom}_{\mathcal{P}_{is}(R)}((L,r),(M,s)) = \begin{cases} \operatorname{Isom}_R(L,M), & r = s, \\ \varnothing, & r \neq s. \end{cases}$$

This category is equipped with tensor products

$$(L,r) \otimes_R (M,s) = (L \otimes_R M, r+s)$$

with commutativity isomorphisms

$$\begin{aligned} \psi\colon (L,r) \otimes_R (M,s) &\xrightarrow{\cong} (M,s) \otimes_R (L,r), \\ \ell \otimes m &\mapsto (-1)^{r\,s}\, m \otimes \ell \end{aligned}$$

for $\ell \in L$, $m \in M$.

The unit object with respect to this product is $\mathbb{1} = (R, 0)$, and for each $(L, r) \in \mathcal{P}_{is}(R)$ the inverse is given by $(L^{-1}, -r)$ where $L^{-1} = \underline{\operatorname{Hom}}_R(L, R)$. The canonical evaluation morphism $L \otimes_R \underline{\operatorname{Hom}}_R(L, R) \to R$ induces an isomorphism

$$(L,r) \otimes_R (L^{-1}, -r) \cong \mathbb{1}.$$

A.1. Definition. We denote by $\mathcal{C}_{is}(R)$ the category whose objects are finitely generated projective R-modules and whose morphisms are isomorphisms. For $A \in \mathcal{C}_{is}(R)$ we define the corresponding determinant as an object in $\mathcal{P}_{is}(R)$ given by

$$\det_R(A) = \Big(\bigwedge_R^{\operatorname{rk}_R A} A, \operatorname{rk}_R A \Big).$$

Here $\operatorname{rk}_R A$ is the rank of A, so that the top exterior power $\bigwedge_R^{\operatorname{rk}_R A} A$ is an invertible R-module.

This gives a functor $\det_R\colon \mathcal{C}_{is}(R) \to \mathcal{P}_{is}(R)$. The main result of [KM1976, Chapter I] is that this construction may be generalized as follows. Let $\mathbf{D}(R)$ be the derived category of the category of R-modules. Recall that a complex $A^\bullet$ is **perfect** if it is quasi-isomorphic to a bounded complex of finitely generated projective R-modules. We denote by $\mathcal{P}arf_{is}(R)$ the subcategory of $\mathbf{D}(R)$ whose objects consist of perfect complexes, and whose morphisms are quasi-isomorphisms of complexes.

A.2. Theorem (Knudsen–Mumford). *The determinant may be extended to perfect complexes of R-modules as follows.*

I) For every ring R there exists a functor

$$\det_R\colon \mathcal{P}arf_{is}(R) \to \mathcal{P}_{is}(R)$$

such that $\det_R(0) = \mathbb{1}$.

*$\mathcal{P}$ for "Picard".

II) For every short exact sequence of complexes in $\mathcal{P}arf_{is}(R)$

$$0 \to A^\bullet \xrightarrow{\alpha} B^\bullet \xrightarrow{\beta} C^\bullet \to 0$$

there exists an isomorphism

$$i_R(\alpha, \beta)\colon \det_R A^\bullet \otimes_R \det_R C^\bullet \xrightarrow{\cong} \det_R B^\bullet.$$

In particular, for the short exact sequence

$$\begin{gathered} 0 \longrightarrow A^\bullet \xrightarrow{id} A^\bullet \longrightarrow 0^\bullet \longrightarrow 0 \\ \textit{(resp. } 0 \longrightarrow 0^\bullet \longrightarrow A^\bullet \xrightarrow{id} A^\bullet \longrightarrow 0 \textit{)} \end{gathered}$$

the isomorphism $i_R(id, 0)$ (resp. $i_R(0, id)$) is the canonical isomorphism

$$\det_R A^\bullet \otimes_R \mathbb{1} \xrightarrow{\cong} \det_R A^\bullet$$

Moreover, the following properties hold.

i) Given an isomorphism of short exact sequences of complexes

$$\begin{tikzcd} 0 \arrow[r] & A^\bullet \arrow[r, "\alpha"] \arrow[d, "u", "\cong"'] & B^\bullet \arrow[r, "\beta"] \arrow[d, "v", "\cong"'] & C^\bullet \arrow[r] \arrow[d, "w", "\cong"'] & 0 \\ 0 \arrow[r] & A'^\bullet \arrow[r, "\alpha'"] & B'^\bullet \arrow[r, "\beta'"] & C'^\bullet \arrow[r] & 0 \end{tikzcd}$$

the diagram

$$\begin{tikzcd} \det_R A^\bullet \otimes_R \det_R C^\bullet \arrow[r, "i^*(\alpha,\beta)", "\cong"'] \arrow[d, "\det_R(u)\otimes\det_R(w)", "\cong"'] & \det_R B^\bullet \arrow[d, "\det_R(v)", "\cong"'] \\ \det_R A'^\bullet \otimes_R \det_R C'^\bullet \arrow[r, "i^*(\alpha',\beta')", "\cong"'] & \det_R B'^\bullet \end{tikzcd}$$

commutes.

ii) Given a commutative 3×3 diagram with rows and columns short exact sequences

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & A^\bullet \arrow[r, "\alpha"] \arrow[d, "u"] & B^\bullet \arrow[r, "\beta"] \arrow[d, "u'"] & C^\bullet \arrow[r] \arrow[d, "u''"] & 0 \\ 0 \arrow[r] & A'^\bullet \arrow[r, "\alpha'"] \arrow[d, "v"] & B'^\bullet \arrow[r, "\beta'"] \arrow[d, "v'"] & C'^\bullet \arrow[r] \arrow[d, "v''"] & 0 \\ 0 \arrow[r] & A''^\bullet \arrow[r, "\alpha''"] \arrow[d] & B''^\bullet \arrow[r, "\beta''"] \arrow[d] & C''^\bullet \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

the diagram

$$\begin{tikzcd} \det_R A^\bullet \otimes_R \det_R C^\bullet \otimes_R \det_R A''^\bullet \otimes_R \det_R C''^\bullet \arrow[r, "i_R(\alpha,\beta)\otimes i_R(\alpha'',\beta'')", "\cong"'] \arrow[d, "id\otimes\psi\otimes id", "\cong"'] & \det_R B^\bullet \otimes_R \det_R B''^\bullet \arrow[dd, "i_R(u',v')", "\cong"'] \\ (\det_R A^\bullet \otimes_R \det_R A''^\bullet) \otimes_R (\det_R C^\bullet \otimes_R \det_R C''^\bullet) \arrow[d, "i_R(u,v)\otimes i_R(u'',v'')", "\cong"'] & \\ \det_R A'^\bullet \otimes_R \det_R C'^\bullet \arrow[r, "i_R(\alpha',\beta')", "\cong"'] & \det_R B'^\bullet \end{tikzcd}$$

commutes.

iii) det *and* i *commute with base change. Namely, given a ring homomorphism* $f\colon R \to S$*, there is a natural isomorphism*

$$\eta_A = \eta_{A^\bullet}(f)\colon\ \det_S(A^\bullet \otimes^{\mathbf{L}}_R S) \xrightarrow{\cong} (\det_R A^\bullet) \otimes_R S,$$

such that for every short exact sequence of complexes

$$0 \to A^\bullet \xrightarrow{\alpha} B^\bullet \xrightarrow{\beta} C^\bullet \to 0$$

the diagram

$$\begin{tikzcd}
\det_S(A^\bullet \otimes^{\mathbf{L}}_R S) \otimes_S \det_S(C^\bullet \otimes^{\mathbf{L}}_R S) \arrow[r, "i_S(\alpha\otimes S,\beta\otimes S)", "\cong"'] \arrow[d, "\cong"', "\eta_A\otimes\eta_C"] & \det_S(B^\bullet \otimes^{\mathbf{L}}_R S) \arrow[d, "\cong"', "\eta_B"] \\
\Big((\det_R A^\bullet)\otimes_R S\Big) \otimes_S \Big((\det_R C^\bullet)\otimes_R S\Big) \arrow[r, "i_R(\alpha,\beta)\otimes S", "\cong"'] & (\det_R B^\bullet)\otimes_R S
\end{tikzcd}$$

commutes. Similarly, there is compatibility with compositions of base changes along $R \xrightarrow{f} S \xrightarrow{g} T$ *(we omit the corresponding commutative diagram).*

A.3. Remark. We refer to [KM1976] for the actual construction. In practice, the following considerations are useful; see [ibid.] for the proofs.

1) If $A^\bullet$ is a bounded complex where each object A^i is perfect (i.e. admits a finite length resolution by finitely generated projective R-modules), then
$$\det_R A^\bullet \cong \bigotimes_{i\in\mathbb{Z}} (\det_R A^i)^{(-1)^i}.$$
In particular, if each A^i is already a finitely generated projective R-module, then $\det_R A^i$ in the above formula is given by A.1.

2) If the cohomology modules $H^i(A^\bullet)$ are perfect, then
$$\det_R A^\bullet \cong \bigotimes_{i\in\mathbb{Z}} (\det_R H^i(A^\bullet))^{(-1)^i}.$$

The determinants also behave well not only with short exact sequences, but with distinguished triangles.

A.4. Proposition ([KM1976, Proposition 7]). *For a distinguished triangle of complexes in* $\mathcal{P}\mathit{arf}_{is}(R)$

$$A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \xrightarrow{w} A^\bullet[1]$$

there is a canonical isomorphism

$$i_R(u,v,w)\colon\ \det_R A^\bullet \otimes_R \det_R C^\bullet \xrightarrow{\cong} \det_R B^\bullet,$$

which is functorial in the following sense: given a (quasi-)isomorphism of distinguished triangles

$$\begin{tikzcd}
A^\bullet \arrow[r, "u"] \arrow[d, "\cong"', "f"] & B^\bullet \arrow[r, "v"] \arrow[d, "\cong"', "g"] & C^\bullet \arrow[r, "w"] \arrow[d, "\cong"', "h"] & A^\bullet[1] \arrow[d, "\cong"', "f[1]"] \\
A'^\bullet \arrow[r, "u"] & B'^\bullet \arrow[r, "v"] & C'^\bullet \arrow[r, "w"] & A'^\bullet[1]
\end{tikzcd}$$

the diagram

$$\begin{tikzcd}
\det_R A^\bullet \otimes_R \det_R C^\bullet \arrow[r, "i_R(u,v,w)", "\cong"'] \arrow[d, "\cong"', "\det_R(f)\otimes\det_R(h)"] & \det_R B^\bullet \arrow[d, "\cong"', "\det_R(g)"] \\
\det_R A'^\bullet \otimes_R \det_R C'^\bullet \arrow[r, "i_R(u',v',w')", "\cong"'] & \det_R B'^\bullet
\end{tikzcd}$$

commutes.

A.5. Remark. In what follows, for $(L, r) \in \mathcal{P}_{is}(R)$ we will forget about r and treat the determinant as an invertible R-module.

A particular very simple case of interest is when all cohomology groups $H^i(A^\bullet)$ are finite; then it is easy to understand what the determinant means.

A.6. Lemma.

1) Let A be a finite abelian group. Then

$$(\det_{\mathbb{Z}} A) \subset (\det_{\mathbb{Z}} A) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \det_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q}) = \det_{\mathbb{Q}}(0) \cong \mathbb{Q}$$

corresponds to the fractional ideal $\frac{1}{\# A}\mathbb{Z} \subset \mathbb{Q}$.

2) In general, let $A^\bullet$ be a perfect complex of abelian groups such that the cohomology groups $H^i(A^\bullet)$ are all finite. Then $\det_{\mathbb{Z}} A^\bullet$ corresponds to the fractional ideal $\frac{1}{m}\mathbb{Z} \subset \mathbb{Q}$, where

$$m = \prod_{i \in \mathbb{Z}} |H^i(A^\bullet)|^{(-1)^i}.$$

Proof. Since $\det_{\mathbb{Z}}(A \oplus B) \cong \det_{\mathbb{Z}} A \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} B$, in part 1) it would be enough to consider the case of a cyclic group $A = \mathbb{Z}/m\mathbb{Z}$. Then we have a quasi-isomorphism of complexes

$$\mathbb{Z}/m\mathbb{Z}[0] \cong \Big[\underset{\text{deg. } -1}{m\mathbb{Z}} \hookrightarrow \underset{\text{deg. } 0}{\mathbb{Z}}\Big].$$

Therefore,

$$\det_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z} \otimes_{\mathbb{Z}} (m\mathbb{Z})^{-1} \cong (m\mathbb{Z})^{-1},$$

which corresponds to $\frac{1}{m}\mathbb{Z}$ inside $\mathbb{Q}$. Part 2) follows immediately from 1) using the isomorphism

$$\det_{\mathbb{Z}} A^\bullet \cong \bigotimes_{i \in \mathbb{Z}} (\det_{\mathbb{Z}} H^i(A^\bullet))^{(-1)^i}. \qquad \square$$

A.7. Remark. The above argument works in a more general setting, assuming R is a regular noetherian ring and A is a finitely generated torsion R-module (replacing $\mathbb{Q}$ with the total quotient field $Q(R)$).

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