WEIL-ÉTALE COHOMOLOGY OF ARITHMETIC SCHEMES

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OUTLINE

I. **Motivation**: arithmetic zeta functions, special values, and their cohomological interpretation.
II. **Lichtenbaum's Weil-étale program**: ideas and known results.
III. **Constructions and conjectures for n < 0** (my work).
IV. **Some new unconditional results**: one-dimensional and cellular schemes.
V. **Some questions for the future**.

PART I.

(MOTIVIC) MOTIVATION

ARITHMETIC SCHEMES AND THEIR ZETA FUNCTIONS

- **Arithmetic scheme** X:
  separated, of finite type over $\operatorname{Spec}\mathbb{Z}$.
- **Zeta function**:

$$X \rightsquigarrow \zeta(X,s) = \prod_{\substack{x\in X \\ \text{closed}}} \frac{1}{1-|\kappa(x)|^{-s}}$$

- Convergence for $s > \dim X$.
- Big conjecture: meromorphic continuation to $s \in \mathbb{C}$.
- Big conjecture: functional equation
  $\zeta(X,s) \leftrightarrow \zeta(X, \dim X - s)$.

ARITHMETIC VS. GEOMETRY

arithmetic

$$\begin{array}{c} X \\ \downarrow{\scriptstyle \text{f.t.}} \\ \operatorname{Spec}\mathbb{Z} \end{array}$$

geometry

$$\begin{array}{ccc} X & & \\ \downarrow{\scriptstyle \text{f.t.}} & \searrow & \\ \operatorname{Spec} k & \dashrightarrow & \\ & & \operatorname{Spec}\mathbb{Z} \end{array}$$

- Both of the worlds: varieties $X/\mathbb{F}_q$.
- Mixed characteristic: usually harder.

DEDEKIND ZETA FUNCTION (XIX CENTURY)

- **Number field**: finite extension $F/\mathbb{Q}$.
- **Ring of integers**: the "integral model":

$$\begin{array}{ccc} \mathcal{O}_F & \subset & F \\ {\scriptstyle \mathrm{rk}=[F:\mathbb{Q}]}\Big| & & \Big| \\ \mathbb{Z} & \subset & \mathbb{Q} \end{array}$$

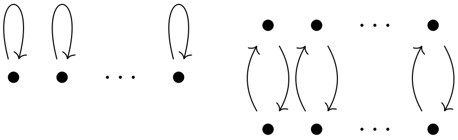
- $\dim \mathcal{O}_F = 1$.
- **Dedekind zeta function**:

$$\begin{aligned} \zeta_F(s) &:= \zeta(\operatorname{Spec} \mathcal{O}_F, s) \\ &= \prod_{\mathfrak{m} \subset \mathcal{O}_F} \frac{1}{1 - |\mathcal{O}_F/\mathfrak{m}|^{-s}} \\ &= \sum_{0 \neq \mathfrak{a} \subseteq \mathcal{O}_F} \frac{1}{|\mathcal{O}_F/\mathfrak{a}|^s}. \quad \text{(Euler)} \end{aligned}$$

- Primordial example: $\zeta_{\mathbb{Q}}(s) = \zeta(\operatorname{Spec} \mathbb{Z}, s) = \zeta(s)$.

MORE ON NUMBER FIELDS ($X = \operatorname{Spec} \mathcal{O}_F$)

- **Real and complex places**. Consider $\operatorname{Gal}(\mathbb{C}/\mathbb{R}) \curvearrowright X(\mathbb{C})$:



- $r_1 = |X(\mathbb{R})|$ and $|X(\mathbb{C})| = r_1 + 2r_2$.
- **Abelian number fields**: $F/\mathbb{Q}$ Galois, with $\operatorname{Gal}(F/\mathbb{Q})$ abelian. Usually easier. Reason: **Kronecker–Weber** and good understanding of cyclotomic fields.

$$F/\mathbb{Q} \text{ abelian} \iff F \subseteq \mathbb{Q}(e^{\frac{2\pi\sqrt{-1}}{N}}) \text{ for some } N.$$

- **Nonmaximal orders**: $\mathcal{O} \subsetneq \mathcal{O}_F$ s.t. $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q} \cong F$.
  $\operatorname{Spec} \mathcal{O}$ is not regular.
  Example: $\mathbb{Z}[\sqrt{5}] \subsetneq \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right] \subset \mathbb{Q}(\sqrt{5})$.

HASSE-WEIL ZETA FUNCTION (XX CENTURY)

- $X/\mathbb{F}_q$ variety over finite field.
- $Z(X,t) := \exp\left(\sum_{k\geq 1} \frac{|X(\mathbb{F}_{q^k})|}{k} t^k\right)$.
- $\zeta(X,s) = Z(X, q^{-s})$.
- Weil conjectures (1949).
- Dwork: $Z(X,t) \in \mathbb{Q}(t)$.
- Full proofs of Weil conjectures: 60s through mid 70s (Grothendieck, ..., Deligne)

SPECIAL VALUES

- Fix $n \in \mathbb{Z}$.
- Assume analytic continuation at $s = n$.
- **Vanishing order** at $s = n$:

$$d_n := \operatorname{ord}_{s=n} \zeta(X, s).$$

- **Special value** at $s = n$:

$$\zeta^*(X, n) := \lim_{s \to n} (s - n)^{-d_n} \zeta(X, s).$$

CLASS NUMBER FORMULA (DIRICHLET)

- Consider $s = n = 0$.
- $\operatorname{ord}_{s=0} \zeta_F(s) = r_1 + r_2 - 1 = \operatorname{rk} \mathcal{O}_F^\times$ (Dirichlet's unit theorem).
- $\zeta_F^*(0) = -\frac{|\operatorname{Pic}(\mathcal{O}_F)|}{|(\mathcal{O}_F)_{tors}^\times|} R_F$.
- R_F — regulator $\cong$ covolume of a canonical embedding $\mathcal{O}_F^\times \hookrightarrow \mathbb{R}^{r_1+r_2-1}$ (cf. unit theorem).
- Similar for smooth projective curves $X/\mathbb{F}_q$: $\operatorname{ord}_{s=0} \zeta(X, s) = -1$ and $\zeta^*(X, 0) = \frac{|\operatorname{Pic}^0(X)|}{|\mathbb{F}_q^\times|}$.
- Generalizations to other $s = n \in \mathbb{Z}$?

ÉTALE MOTIVIC COHOMOLOGY

- Lichtenbaum, 1984: hypothetical complexes of sheaves on $X_{ét}$ "responsible" for special values.
- Bloch, 1986: cycle complexes / higher Chow groups.
- Étale version: complex of sheaves $\mathbb{Z}(n)$ on $X_{ét}$.
- Levine, Geisser, ...: works fine for $X/\operatorname{Spec}\mathbb{Z}$.
- Few explicit calculations.
- Not even finite generation.

BOREL-MOORE VERSION

- For those working with $\mathbb{Z}(n)$...
- Complex of sheaves $\mathbb{Z}^c(n)$ on $X_{\acute{e}t}$:

$$z_n(X,i) = \mathbb{Z}\left\langle Z \overset{\dim n+i}{\subset} X \times \Delta^i \;\middle|\; \begin{array}{c}\text{cl. integral subsch.}\\ \text{proper int. with faces}\end{array} \right\rangle,$$

$$\mathbb{Z}^c(n) := U \rightsquigarrow z_n(U, -\bullet)[2n].$$

- **Borel–Moore** behavior: triangles for $Z \not\hookrightarrow X \hookleftarrow U$

$$R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n)) \to R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)) \to R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)) \to [+1].$$

- For X proper and regular, $d = \dim X$:

$$H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) \cong H^{i+2d}(X_{\acute{e}t}, \mathbb{Z}(d-n)).$$

COHOMOLOGICAL INTERPRETATION OF VANISHING ORDERS

- Consider $F/\mathbb{Q}$ and $n \leq 0$.
- $\approx$ Borel, 1974:

$$d_n = \operatorname{ord}_{s=n} \zeta_F(s) = \operatorname{rk}_{\mathbb{Z}} H^{-1}(\operatorname{Spec} \mathcal{O}_{F,ét}, \mathbb{Z}^c(n))$$
$$= \begin{cases} r_1 + r_2 - 1, & n = 0, \\ r_1 + r_2, & n < 0 \text{ even}, \\ r_1, & n < 0 \text{ odd}. \end{cases}$$

COHOMOLOGICAL INTERPRETATION OF SPECIAL VALUES

- **Conjecture**:

$$\zeta_F^*(n) = \pm \frac{|H^0(X_{\acute{e}t}, \mathbb{Z}^c(n))|}{|H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{tors}|} R_{F,n}.$$

- **Higher regulators**: Borel, Beilinson, ...:

$$R_{F,n} = \operatorname{vol}\operatorname{coker}\Big(\underbrace{H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))}_{\mathrm{rk}_{\mathbb{Z}} = d_n} \to \underbrace{H^1_{\mathcal{D}}(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))}_{\dim_{\mathbb{R}} = d_n}\Big).$$

- **Known for** abelian $F/\mathbb{Q}$, via TNC (Benois, Nguyen Quang Do, Huber, Kings, Flach, ...)
- **Lichtenbaum, 1973**: in terms of $K_i(\mathcal{O}_F)$, for F real ($r_2 = 0$), odd n (hence $R_{F,n} = 1$).

CASE OF VARIETIES $X/\mathbb{F}_q$

- Consider $n < 0$.
- $\mathrm{ord}_{s=n}\,\zeta(X,s) = 0$.
- Assuming the groups $H^i(X_{ét}, \mathbb{Z}^c(n))$ are f.g.,

$$\zeta(X,n) = \pm \prod_{i\in\mathbb{Z}} |H^i(X_{ét}, \mathbb{Z}^c(n))|^{(-1)^i}.$$

- Reason: Grothendieck's trace formula + ϵ.
- Case of $n \geq 0$: more difficult; Milne (1986), ...

PART II.

WEIL-ÉTALE COHOMOLOGY

STRUCTURE OF MOTIVIC COHOMOLOGY FOR $X/\mathbb{Z}$ (LICHTENBAUM)

- Conjecture:

$$H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) = \begin{cases} \text{f.g.}, & i \leq -2n, \\ \text{finite}, & i = -2n+1, \\ \text{cofinite type}, & i \geq -2n+2. \end{cases}$$

- **Cofinite type** = $\mathbb{Q}/\mathbb{Z}$-dual to f.g.
  Manifestation of **arithmetic duality**
  (Artin, Verdier 1964, ...).
- * If $n < 0$, then $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are conjecturally f.g.
- **Beilinson–Soulé conjecture**: $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) = 0$ for $i < -2 \dim X$.
- In general, $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) \neq 0$ for $i \gg 0$.

STRUCTURE OF MOTIVIC COHOMOLOGY FOR $X/\mathbb{F}_q$ (LICHTENBAUM)

- Conjeturally,

$$H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) = \begin{cases} \text{finite}, & i \neq -2n, -2n+2, \\ \text{f.g.}, & i = -2n, \\ \text{cofinite type}, & i = -2n+2. \end{cases}$$

- * if $n < 0$, then $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are conjecturally finite.

WEIL-ÉTALE COHOMOLOGY (LICHTENBAUM)

- Étale motivic cohomology $\rightsquigarrow$ Weil-étale cohomology.
- $H^i_{W,c}(X, \mathbb{Z}(n))$ finitely generated, $= 0$ for $|i| \gg 0$.
- Long exact sequence

$$\cdots \to H^i_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \xrightarrow{\smile\theta} H^{i+1}_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \to \cdots$$

- $H^i_{W,c}(X, \mathbb{Z}(n))$ "encodes" $\operatorname{ord}_{s=n} \zeta(X, s)$ and $\zeta^*(X, n)$.
- Why "Weil-étale"? A construction for $X/\mathbb{F}_q$:

$$R\Gamma(G, R\Gamma(X_{\overline{\mathbb{F}}_q, ét}, \mathbb{Z}^c(n))).$$

  $G \subset \operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ = Weil group, generated by the Frobenius ($\cong \mathbb{Z} \subset \widehat{\mathbb{Z}}$).
- Weil-étale topos?

SOME RESULTS

- A "result" =
  - define $H^i_{W,c}(X, \mathbb{Z}(n))$, assuming Lichtenbaum's conjectures on the structure of $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$,
  - formulate the conjectural relation of $H^i_{W,c}(X, \mathbb{Z}(n))$ to $\operatorname{ord}_{s=n} \zeta(X, s)$ and $\zeta^*(X, n)$,
  - relate to other conjectures, prove some particular cases.
- **Lichtenbaum** (2005): $X/\mathbb{F}_q$.
- **Geisser** (2004–...): $X/\mathbb{F}_q$, possibly singular (*eh*-topology).
- **Lichtenbaum** (2009): $X = \operatorname{Spec} \mathcal{O}_F$.
- **Morin** (2014): $X/\mathbb{Z}$ proper and regular, $n = 0$.
- **Flach, Morin** (2018): ———————, $n \in \mathbb{Z}$.
- **B.** (2020/21): any arithmetic scheme $X/\mathbb{Z}$, $n < 0$.

PART III.

CONSTRUCTIONS AND CONJECTURES FOR $n < 0$

WEIL-ÉTALE COMPLEXES

- $X \to \operatorname{Spec}\mathbb{Z}$ separated, of finite type, $n < 0$.
- Assume a Lichtenbaum-type conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$: $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are fintely generated for all $i \in \mathbb{Z}$.
- There is a complex $R\Gamma_{W,c}(X, \mathbb{Z}(n)) \in \mathcal{D}(\mathbb{Z})$.
- Perfectness of the complex:

$$H^i_{W,c}(X, \mathbb{Z}(n)) := H^i(R\Gamma_{W,c}(X, \mathbb{Z}(n)))$$

are finitely generated, $= 0$ for $i \notin [0, 2\dim X + 1]$.

WEIL-ÉTALE COMPLEXES (CONT.)

- For $X/\mathbb{F}_q$, there is an isomorphism of finite groups

$$\begin{aligned} H^i_{W,c}(X, \mathbb{Z}(n)) &\cong \mathrm{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \\ &\cong H^{i-1}_c(X_{\acute{e}t}, \mathbb{Q}/\mathbb{Z}'(n)), \quad \mathbb{Q}/\mathbb{Z}'(n) = \varinjlim_{p \nmid m} \mu_m^{\otimes n}. \end{aligned}$$

- Splitting with rational / real coefficients:

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \cong \begin{array}{c} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \\ \oplus \\ R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n))[-1] \end{array}$$

- $\mathbb{R}(n) := (2\pi i)^n \, \mathbb{R}$, $G_\mathbb{R} := \mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

PRINCIPAL INGREDIENT

- Arithmetic duality

$$\mathrm{Hom}(\underbrace{H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))}_{\text{f.g.}}, \mathbb{Q}/\mathbb{Z}) \cong \underbrace{\widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}'(n))}_{\text{cofinite type}},$$

- Based on work of Geisser (2010).
- $\mathbb{Z}'(n) = \mathbb{Q}/\mathbb{Z}'(n)[-1] = \bigoplus_p \varinjlim_r j_{p!}\mu_{p^r}^{\otimes n}[-1]$, $j_p \colon X[1/p] \hookrightarrow X$.
- * $\widehat{H}^i_c$ = modified cohomology with compact support, treats $X(\mathbb{R})$.

$$\widehat{H}^i_c(X_{\acute{e}t}, \mathcal{F}^\bullet) \cong H^i_c(X_{\acute{e}t}, \mathcal{F}^\bullet) \quad \text{up to 2-torsion}$$

- Generalization of Artin–Verdier duality for $X = \operatorname{Spec} \mathcal{O}_F$.

REGULATORS

- Assume the fiber $X_{\mathbb{C}}$ is smooth.
- Construction of Kerr–Lewis–Müller-Stach (2006) $\Longrightarrow$

$$Reg\colon R\Gamma(X_{\acute{e}t}, \mathbb{R}^c(n)) \to R\mathrm{Hom}(R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)), \mathbb{R}[1]).$$

- * the target is not Deligne–Beilinson cohomology, but simply $H^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))^\vee$, since $n < 0$.
- Conjecture $\mathbf{B}(X, n)$ (Beilinson):

$$Reg^\vee\colon R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \to R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})$$

is a quasi-isomorphism.

VANISHING ORDER CONJECTURE

- **VO**(X,n): assuming $\mathbf{L}^c(X,n)$,

$$\mathrm{ord}_{s=n}\,\zeta(X,s) = \sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot \mathrm{rk}_{\mathbb{Z}}\, H^i_{W,c}(X,\mathbb{Z}(n)). \qquad (*)$$

- Assuming $\mathbf{B}(X,n)$,

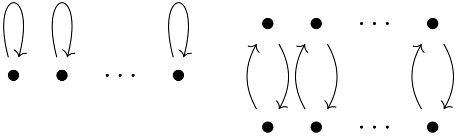
$$\mathrm{ord}_{s=n}\,\zeta(X,s) = \sum_{i\in\mathbb{Z}} (-1)^i \dim_{\mathbb{R}} H^i_c(X(\mathbb{C}),\mathbb{R}(n))^{G_{\mathbb{R}}} \qquad (**)$$

$$= \sum_{i\in\mathbb{Z}} (-1)^{i+1} \mathrm{rk}_{\mathbb{Z}}\, H^i(X_{\acute{e}t},\mathbb{Z}^c(n)). \qquad (***)$$

- (**) agrees with the (conjectural) functional equation.
  For $n < 0$ zeros and poles come from the Γ-factors.
- (***) agrees with a conjecture of Soulé (1984).

TOY EXAMPLE

- For $X = \operatorname{Spec} \mathcal{O}_F$ consider $X(\mathbb{C})$:



r_1 points $\qquad$ $2\, r_2$ points

- Complex $R\Gamma_c(X(\mathbb{C}), \mathbb{R}(n))$:

$$\mathbb{R}(n)^{\oplus r_1} \oplus (\mathbb{R}(n) \oplus \mathbb{R}(n))^{r_2},$$

$G_{\mathbb{R}}$-action by $z \mapsto \bar{z}$ resp. $(z, w) \mapsto (\overline{w}, \bar{z})$.

- $\operatorname{ord}_{s=n} \zeta_F(s) = \dim_{\mathbb{R}} H^0_c(X(\mathbb{C}), \mathbb{R}(n))^{G_{\mathbb{R}}} =$ $\operatorname{rk}_{\mathbb{Z}} H^{-1}_{\acute{e}t}(X, \mathbb{Z}^c(n)) = \left\{ \begin{array}{ll} r_1 + r_2, & n \text{ even}, \\ r_2, & n \text{ odd}. \end{array} \right\}$

DETERMINANTS OF COMPLEXES

- For projective f.g. modules:
  $\det_R P := \bigwedge^{\operatorname{rk} P} P$
  (invertible = projective of rk 1).
- Functor
  $$\begin{pmatrix} \text{projective f.g. modules,} \\ \text{isomorphisms} \end{pmatrix} \rightsquigarrow \begin{pmatrix} \text{invertible modules,} \\ \text{isomorphisms} \end{pmatrix}.$$
- Knudsen, Mumford, 1976: extension
  $$\begin{pmatrix} \text{perfect complexes,} \\ \text{quasi-isomorphisms} \end{pmatrix} \rightsquigarrow \begin{pmatrix} \text{invertible modules,} \\ \text{isomorphisms} \end{pmatrix}.$$
- $\det_R A^\bullet \cong \bigotimes_{i\in\mathbb{Z}}(\det_R H^i(A^\bullet))^{(-1)^i}$, $\det_R 0 \cong R$.
- Compatible with base change.

TRIVIALIZATION MORPHISM

- Quasi-isomorphism of complexes, assuming $\mathbf{B}(X,n)$:

$$\begin{array}{c} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-2] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \end{array}$$

$$\text{* } \det_R(A^\bullet \oplus A^\bullet[1]) \cong R$$

$$\cong \Big\downarrow Reg^\vee[-1] \oplus id$$

$$\begin{array}{c} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \end{array} \xrightarrow[\cong]{\text{split}} R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R}$$

- Canonical isomorphism of determinants:

$$\lambda\colon \mathbb{R} \xrightarrow{\cong} \det_{\mathbb{R}}\Big(R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R}\Big) \cong \Big(\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n))\Big) \otimes \mathbb{R}.$$

SPECIAL VALUE CONJECTURE

- Consider

$$\lambda\colon \mathbb{R} \xrightarrow{\cong} \underbrace{(\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)))}_{\mathbb{Z}\text{-mod of rk } 1} \otimes \mathbb{R}.$$

- Assume
  - $\mathbf{L}^c(X_{ét}, n)$: finite generation of $H^i(X_{ét}, \mathbb{Z}^c(n))$,
  - smooth fiber $X_{\mathbb{C}}$,
  - $\mathbf{B}(X, n)$: regulator conjecture,
  - meromorphic continuation around $s = n < 0$.
- $\mathbf{C}(X, n)$: the special value at $s = n < 0$ is determined up to sign by

$$\lambda(\zeta^*(X, n)^{-1}) \cdot \mathbb{Z} = \det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)).$$

CASE OF VARIETIES $X/\mathbb{F}_q$

- Assuming $\mathbf{L}^c(X_{\acute{e}t}, n)$, the conjecture $\mathbf{C}(X, n)$ is equivalent to

$$\zeta(X, n) = \pm \prod_{i \in \mathbb{Z}} |H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))|^{(-1)^i}.$$

- $X(\mathbb{C}) = \emptyset$, there's no regulator.
- A singular example: nodal cubic $X = \mathbb{P}^1_{\mathbb{F}_q}/(0 \sim 1)$.

$$H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)) = \mathbb{Z}/(q^{1-n} - 1),$$
$$H^{0,1}(X_{\acute{e}t}, \mathbb{Z}^c(n)) = \mathbb{Z}/(q^{-n} - 1).$$

$$\zeta(X, s) = \frac{1}{1 - q^{1-s}}.$$

- $\mathbf{L}^c(X_{\acute{e}t}, n) \Longrightarrow \mathbf{C}(X, n)$ for any $n < 0$ and $X/\mathbb{F}_q$.

COMPATIBILITIES

- **Disjoint unions**: if $X = \coprod_{1\le i\le r} X_i$, then

$$\zeta(X,s) = \prod_{1\le i\le r} \zeta(X_i,s).$$

- Accordingly,

$$\mathbf{VO}(X,n) \iff \mathbf{VO}(X_i,n) \text{ for all } i,$$
$$\mathbf{C}(X,n) \iff \mathbf{C}(X_i,n) \text{ for all } i.$$

- **Closed-open decompositions**: for $Z \not\hookrightarrow X \hookleftarrow U$,

$$\zeta(X,s) = \zeta(Z,s)\cdot\zeta(U,s).$$

- Two out of three conjectures $\mathbf{VO}(X,n)$, $\mathbf{VO}(Z,n)$, $\mathbf{VO}(U,n)$ (resp. $\mathbf{C}(X,n)$, $\mathbf{C}(Z,n)$, $\mathbf{C}(U,n)$) imply the third.
- **Affine bundles**: $\zeta(\mathbb{A}^r_X,s) = \zeta(X,s-r)$.
- $\mathbf{VO}(\mathbb{A}^r_X,n) \iff \mathbf{VO}(X,n-r)$, $\mathbf{C}(\mathbb{A}^r_X,n) \iff \mathbf{C}(X,n-r)$.

PART IV.

NEW UNCONDITIONAL RESULTS

ONE-DIMENSIONAL SCHEMES

- Let B be a 1-dimensional arithmetic scheme.
- We say it's **abelian** if for each generic point $\eta \in B$ holds
  a) $\operatorname{char} \kappa(\eta) = p > 0$, or
  b) $\operatorname{char} \kappa(\eta) = 0$ and $\kappa(\eta)/\mathbb{Q}$ is abelian.
- **Theorem** (B.): **VO**(B, n) and **C**(B, n) hold unconditionally for any $n < 0$ and abelian 1-dimensional B.
- **Proof idea**: the cases of $B = \operatorname{Spec} \mathcal{O}_F$ and $B/\mathbb{F}_q$ are known. Use compatibilities and proceed by dévissage.

MOTIVIC COHOMOLOGY FOR ONE-DIMENSIONAL B

$$H^i(B_{\acute{e}t}, \mathbb{Z}^c(n)) \cong \begin{cases} 0, & i < -1, \\ \text{f.g. of rk } d_n, & i = -1, \\ \text{finite}, & i = 0, 1, \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus |X(\mathbb{R})|}, & i \geq 2,\ i \not\equiv n\ (2), \\ 0, & i \geq 2,\ i \equiv n\ (2). \end{cases}$$

- Arithmetically interesting part concentrated in $i = -1, 0, +1$.
- Finite 2-torsion for $i \geq 2$ comes from $X(\mathbb{R})$.

WEIL-ÉTALE COHOMOLOGY FOR ONE-DIMENSIONAL B

- $H^i_{W,c}(B, \mathbb{Z}(n)) = 0$ for $i \neq 1, 2, 3$.
- $H^1_{W,c}(B, \mathbb{Z}(n)) \cong$ $\underbrace{H^0_c(G_\mathbb{R}, B(\mathbb{C}), \mathbb{Z}(n))}_{\cong \mathbb{Z}^{\oplus d_n}} \oplus \mathrm{Hom}(\underbrace{H^1(B_{\acute{e}t}, \mathbb{Z}^c(n))}_{\text{finite}}, \mathbb{Q}/\mathbb{Z})$ (more or less, up to finite 2-tosion).
- $H^2_{W,c}(B, \mathbb{Z}(n)) \cong$ $\underbrace{\mathrm{Hom}(H^{-1}(B_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z})}_{\cong \mathbb{Z}^{\oplus d_n}} \oplus \mathrm{Hom}(\underbrace{H^0(B_{\acute{e}t}, \mathbb{Z}^c(n))}_{\text{finite}}, \mathbb{Q}/\mathbb{Z})$.
- $H^3_{W,c}(B, \mathbb{Z}(n)) \cong \mathrm{Hom}(H^{-1}(B_{\acute{e}t}, \mathbb{Z}^c(n))_{tors}, \mathbb{Q}/\mathbb{Z})$.

EXPLICIT FORMULA

$$\zeta^*(B,n) = \pm 2^\delta \frac{|H^0(B_{ét}, \mathbb{Z}^c(n)|}{|H^{-1}(B_{ét}, \mathbb{Z}^c(n))_{tors}| \cdot |H^1(B_{ét}, \mathbb{Z}^c(n))|} R_{B,n};$$

$$\delta = \begin{cases} |B(\mathbb{R})|, & n \text{ even}, \\ 0, & n \text{ odd}; \end{cases}$$

$$R_{B,n} = \text{regulator on } H^{-1}(B_{ét}, \mathbb{Z}^c(n)).$$

- **Theorem** (B.): this is true for abelian B and $n < 0$.
- **Conjecture**: should be true for nonabelian B.

CELLULAR SCHEMES

- **Cellular** scheme $X \to B$: admits filtration by closed subschemes

$$X = Z_N \supseteq Z_{N-1} \supseteq \cdots \supseteq Z_0 \supseteq Z_{-1} = \emptyset,$$

  with $Z_i \setminus Z_{i-1} \cong \coprod_j \mathbb{A}_B^{r_{i_j}}$
- **Theorem** (B.): Given X cellular over a 1-dim abelian base B, with smooth fiber $X_{\mathbb{C}}$, the conjectures **VO**(X, n) and **C**(X, n) hold unconditionally for all $n < 0$.
- **Proof idea**: compatibilities and dévissage.

PART V.

SOME QUESTIONS

SOME QUESTIONS FOR THE FUTURE

- What to do for $n \geq 0$ and non-regular X?
  Geisser (2006): case of singular $X/\mathbb{F}_q$.
  Mixed characteristic? Already interesting case:
  nonmaximal orders $X = \operatorname{Spec} \mathcal{O}$.
- The regulator of Kerr–Lewis–Müller-Stach is defined for smooth $X_{\mathbb{C}}$. How to extend it to the non-smooth case?
- When the comparison makes sense, $\mathbf{C}(X, n)$ is equivalent to TNC. What is the equivariant refinement, equivalent to ETNC?
- More canonical and functorial construction of Weil-étale complexes $R\Gamma_{W,c}(X, \mathbb{Z}(n))$.
- ...

THANK YOU FOR YOUR ATTENTION!