Bottom Lines

(an excerpt from “Indiscrete Thoughts” by Gian-Carlo Rota)

How do mathematicians get to know each other? Professional psychologists do not seem to have studied this question; I will try out an amateur theory. When two mathematicians meet and feel out each other’s knowledge of mathematics, what they are really doing is finding out what each other’s bottom line is. It might be interesting to give a precise definition of a bottom line; in the absence of a definition, we will give some typical examples.

To the algebraic geometers of the sixties, the bottom line was the proof of the Weil conjectures. To generations of German algebraists, from Dirichlet to Hecke and Emil Artin, the bottom line was the theory of algebraic numbers. To the Princeton topologists of the fifties, sixties, and seventies, the bottom line was homotopy. To the functional analysts of Yale and Chicago, the bottom line was the spectrum. To combinatorialists, the bottom lines are the Yang-Baxter equation, the representation theory of the classical groups, and the Schensted algorithm. To some algebraists and combinatorialists of the next ten or so years, the bottom line may be elimination theory.

I will shamelessly tell you what my bottom line is. It is placing balls into boxes, or, as Florence Nightingale David put it with exquisite tact in her book Combinatorial Chance, it is the theory of distribution and occupancy.

We resort to the bottom line when we are asked to write a letter of support for some colleague. If the other mathematician’s bottom line is agreeable with our own, then our letter is more likely to be positive. If our bottom lines disagree, then our letter is likely to be restrained.

The most striking example of mismatch of bottom lines was told to me by Erdös. When David Hilbert, then a professor at the University of Königsberg, was being considered for a professorship at Göttingen, the Prussian ministry asked Professor Frobenius to write a letter in support of Hilbert’s candidacy. Here is what Frobenius wrote: “He is rather a good mathematician, but he will never be as good as Schottky.”

Allow me to tell you two more personal stories. In 1957, in my first year as an instructor in Cambridge, I often had lunch with Oscar Zariski, who liked to practice his Italian. One day while we were sitting in the main room of the Harvard Faculty Club he peered at me, fork in hand and said, loudly enough for everyone to hear, “Remember! Whatever happens in mathematics happens in algebraic geometry first!” Algebraic geometry has been the bottom line of mathematics for almost one hundred years; but perhaps times are changing.

The second story is more somber. One day, in my first year as an assistant professor at MIT, while walking down one of the long corridors, I met Professor Z, a respected senior mathematician with a solid international reputation. He stared at me and shouted, “Admit it! All lattice theory is trivial!” I did not have the presence to answer that yon Neumann’s work in lattice theory is deeper than anything Professor Z has done in mathematics.

Those who have reached a certain age remember the visceral and widespread hatred of lattice theory from around 1940 to 1979; this has not completely disappeared. Such an intense and unusual disliking for an entire field cannot be simply attributed to personality clashes. It is more likely to be explained by pinpointing certain abysmal differences among the bottom lines of the mathematicians of the time. If we begin such a search, we are likely to conclude that the field normally classified as algebra really consists of two quite separate fields. Let us call them Algebra One and Algebra Two for want of a better language.

Algebra One is the algebra whose bottom lines are algebraic geometry or algebraic number theory. Algebra One has by far a better pedigree than Algebra Two, and has reached a high degree of sophistication and breadth. Commutative algebra, homological algebra, and the more recent speculations with categories and topoi are exquisite products of Algebra One. It is not infrequent to meet two specialists in Algebra One who cannot talk to each other since the subject is so vast. Despite repeated and dire predictions of its demise, Algebra One keeps going strong.

Algebra Two has had a more accidented history. It can be traced back to George Boole, who was the initiator of three well-known branches of Algebra Two: Boolean algebra, the operational calculus that views the derivative as the operator D, on which Boole wrote two books of great beauty, and finally, invariant theory, which Boole initiated by remarking the invariance of the discriminant of a quadratic form.

Roughly speaking, between 1850 and 1950 Algebra Two was preferred by the British and the Italians, whereas Algebra One was once a German and lately a French preserve. Capelli and Young's bottom lines were firmly in Algebra Two, whereas Kronecker, Hecke, and Emil Artin are champions of Algebra One.

In the beginning Algebra Two was largely cultivated by invariant theorists. Their objective was to develop a notation suitable to describe geometric phenomena which is independent of the choice of a coordinate system. In pursuing this objective, the invariant theorists of the nineteenth century were led to develop explicit algorithms and combinatorial methods. The first combinatorialists, MacMahon, Hammond, Brioschi, Trudi, Sylvester, were invariant theorists. One of the first papers in graph theory, in which the Petersen graph is introduced, was motivated by a problem in invariant theory. Clifford's ideal for invariant theory was to reduce the computation of invariants to the theory of graphs.

The best known representative of Algebra Two in the nineteenth century is Paul Gordan. He was a German, perhaps the exception that tests our rule. He contributed a constructive proof of the finite generation of the ring of invariants of binary forms which has never been improved upon, and which foreshadows current techniques of Hopf algebra. He also published in 1870 the fundamental results of linear programming, a discovery for which he has never been given proper credit. Despite his achievements, Paul Gordan was never fully accepted by specialists in Algebra One. "Er war ein Algorithmiker!"[1] said Hilbert when Gordan died.

Gordan's student Emmy Noether became an ardent apostle of Algebra One;

[1] "He was an algorithmist!"

similarly, van der Waerden, a student of General Weitzenböck, an Algebra Two hero, intensely disliked Algebra Two throughout his career. In the thirties, Algebra Two was enriched by lattice theory and by the universal algebra of Philip Hall and his student Garrett Birkhoff.

Algebra Two has always had a harder time. You won't find lattices, exterior algebra, or even a mention of tensors in any of the editions of van der Waerden's Modern Algebra. G. H. Hardy subtly condemned Algebra Two in England in the latter half of the nineteenth century with the exclamation, "Too much $f(D)$!" G. H. Hardy must be turning in his grave now.

Algebra Two has recently come of age. In the last twenty years or so, it has blossomed and acquired a name of its own: algebraic combinatorics. Algebraic combinatorics, after a tortuous history, has at last found its own bottom line, together with a firm place in the mathematics of our time.