Zeta-values of arithmetic schemes at negative integers and Weil-étale cohomology

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0 Motivation

Let us consider an **arithmetic scheme** X:

$$\begin{array}{c} X \\ \downarrow {\scriptstyle \text{separated, of finite type}} \\ \operatorname{Spec}\mathbb{Z} \end{array}$$

We may associate to it the corresponding **zeta function**, which is defined by the Euler product

$$\zeta(X,s) := \prod_{x \in X_0} \frac{1}{1 - N(x)^{-s}},$$

where X_0 denotes the set of closed points of X, and $N(x)$ is the cardinality of the residue field of x. For instance, if $X = \operatorname{Spec}\mathbb{Z}$, we recover the usual Riemann zeta function $\zeta(s)$; more generally, if $X = \operatorname{Spec} O_F$ is the spectrum of a number ring, we obtain the Dedekind zeta function $\zeta_F(s)$. The above product converges for $\operatorname{Re} s > \dim X$, and from now on I will also make the following assumption.

Conjecture. *$\zeta(X,s)$ has a meromorphic continuation to the whole complex plane, which we also denote by $\zeta(X,s)$.*

For each integer $n \in \mathbb{Z}$, one might ask about the following two quantities:

1) $d_n := \operatorname{ord}_{s=n} \zeta(X,s) :=$ the vanishing order at $s = n$;

2) the corresponding **special value**, i.e. the leading term of the Taylor expansion at $s = n$:

$$\zeta^*(X,n) := \lim_{s \to n} (s-n)^{-d_n} \, \zeta(X,s).$$

Cohomological interpretation of special values may be traced to 1839 when Dicichlet published the **class number formula**. The Dedekind zeta function $\zeta_F(s)$ has a zero of order $d_0 = r_1 + r_2 - 1$ at $s = 0$, where r_1 and $2\,r_2$ is the number of real and complex places of F, and the corresponding special value is

$$\zeta_F^*(0) = -\frac{h_F}{\# \mu_F} \, R_F,$$

where h_F is the class number, $\mu_F = (O_F^\times)_{tors}$ is the group of roots of unity in F, and R_F is the regulator. The above formula may be written as

$$\zeta^*(\operatorname{Spec} O_F, 0) = -\frac{\# H^1(\operatorname{Spec} O_F, \mathbb{G}_m)}{\# H^0(\operatorname{Spec} O_F, \mathbb{G}_m)_{tors}} \, R_F = -\frac{\# H^0(\operatorname{Spec} O_F, \mathbb{Z}^c(0))}{\# H^{-1}(\operatorname{Spec} O_F, \mathbb{Z}^c(0))_{tors}} \, R_F,$$

where $\mathbb{Z}^c(0) \cong \mathbb{G}_m[1]$ (see below).

In this sense, the first results about finite generation of certain motivic cohomology groups were obtained in the XIX century:

- finiteness of the class group is finiteness of $\# H^0(\operatorname{Spec} \mathcal{O}_F, \mathbb{Z}^c(0))$;
- Dirichlet's unit theorem says that $H^{-1}(\operatorname{Spec} \mathcal{O}_F, \mathbb{Z}^c(0))$ is a group of finite rank $r_1 + r_2 - 1$.

Here I won't get into the details about the classical conjectures of Lichtenbaum, Beĭlinson, and others; instead I refer to the survey [Kah2005]. More recently, Lichtenbaum envisioned the existence of certain cohomology theory, named **Weil-étale cohomology**, that (conjecturally) encodes the information about vanishing orders and special values of $\zeta(X, s)$. Here is a very brief history of the subject.

- Lichtenbaum first studied Weil-étale cohomology for varieties over finite fields in [Lic2005]. Further results were obtained by Thomas Geisser in [Gei2004].
- In [Lic2009] Lichtenbaum considered the case of number rings $X = \operatorname{Spec} \mathcal{O}_F$ and $n = 0$.
- Baptiste Morin constructed in [Mor2014] Weil-étale cohomology for the case of regular, proper X and $n = 0$.
- Matthias Flach and Baptiste Morin generalized this in [FM2016] to all $n \in \mathbb{Z}$, again for regular, proper X.

I am investigating the following situation.

My goal. *Construct and study Weil-étale cohomology for an arbitrary arithmetic scheme X and $n < 0$.*

So from now on, X will denote any arithmetic scheme and n will denote a *strictly negative* integer. Removing the assumptions on X in theory should make everything harder, but at the same time, restricting the attention to the case of $n < 0$ simplifies many things. I am following the ideas of Flach and Morin, and in particular, when X is regular and proper, the constructions and conjectures that I am going to describe coincide with theirs. Not out of immodesty, but due to the lack of time, I will focus on my case and outline the involved tools and definitions.

1 Motivic cohomology

There are several constructions of motivic cohomology. The one that is suitable for arithmetic schemes originates from the seminal paper of Spencer Bloch on higher Chow groups [Blo1986]. Bloch's ideas have been further developed by Marc Levine and other mathematicians, and the corresponding techniques have been also applied to arithmetic schemes (see Geisser's survey [Gei2005]).

Geisser in [Gei2010] introduced **dualizing cycle complexes**, which is a certain variation of Bloch's cycle complexes:

$$(X, n) \rightsquigarrow \mathbb{Z}^c(n), \text{ a complex of sheaves on } X_{\acute{e}t}.$$

For those familiar with motivic cohomology, if X is an equidimensional scheme of dimension d, then

$$\mathbb{Z}^c(n) = \mathbb{Z}(d-n)[2d],$$

where the right hand side is the sheaf defined from Bloch's cycle complex. The same relation would hold with the motivic complex of Voevodsky, if we work with smooth schemes over a field. This is not the case of our interest, however.

Another important thing to keep in mind is that the (hyper)cohomology of $\mathbb{Z}^c(n)$ behaves very much like **Borel–Moore homology** in the topological setting. Namely, if X is a locally compact topological space,

$U \subset X$ is its open subspace, and $Z := X \setminus U$ is the corresponding closed complement, in such a case I will say that we have an **open-closed decomposition**

$$U \hookrightarrow X \leftarrow Z$$

This gives a distinguished triangle in the derived category of abelian groups

$$R\Gamma_{BM}(Z, \mathbb{Z}) \to R\Gamma_{BM}(X, \mathbb{Z}) \to R\Gamma_{BM}(U, \mathbb{Z}) \to \cdots [1]$$

simply because by definition (well, not the original one of Borel and Moore, but the one of Verdier),

$$R\Gamma_{BM}(X, \mathbb{Z}) := R\Gamma(X, p^! \underline{\mathbb{Z}}) \cong R\mathrm{Hom}(R\Gamma_c(X, \mathbb{Z}), \mathbb{Z}),$$

where $p\colon X \to *$ is the projection to a point. So the above distinguished triangle is nothing more than the Verdier dual of the distinguished triangle for cohomology with compact support

$$R\Gamma_c(U, \mathbb{Z}) \to R\Gamma_c(X, \mathbb{Z}) \to R\Gamma_c(Z, \mathbb{Z}) \to \cdots [1]$$

Similarly, for Geisser's complexes $\mathbb{Z}^c(n)$, an open-closed decomposition of schemes gives a distinguished triangle

$$R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n)) \to R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)) \to R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)) \to \cdots [1]$$

The intuition behind this is that initially, $\mathbb{Z}^c(n)$ has an ad hoc definition as a cycle complex, but then it is possible to identify it as a dualizing complex in certain arithmetic contexts. This is what Geisser does in [Gei2010].

In general, not much is known about the cohomology of $\mathbb{Z}^c(n)$, and to proceed, we need to make the following assumption.

Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$. *The (hyper)cohomology groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are finitely generated.*

2 Weil-étale complexes

I don't have enough time to enter in all the gory details of the construction of Weil-étale complexes (for this, I refer to the appendix), so let me state what kind of object it is.

Output of the construction. *Assume that the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$ holds.*

1) *There exists a perfect object in the derived category of abelian groups*

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)),$$

*which we call **Weil-étale cohomology with compact support**. That is, the corresponding cohomology groups*

$$H^i_{W,c}(X, \mathbb{Z}(n)) := H^i(R\Gamma_{W,c}(X, \mathbb{Z}(n)))$$

are finitely generated and vanish for almost all i.

2) *After tensoring with $\mathbb{R}$, this complex splits (non-canonically) as*

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \cong R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \oplus R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R})[-1].$$

I already briefly explained what is $R\Gamma(X_{ét}, \mathbb{Z}^c(n))$. As for the complex $R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})$, it stays for the **$G_\mathbb{R}$-equivariant cohomology with compact support** of the space of complex points $X(\mathbb{C})$, where $G_\mathbb{R} := \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Namely, $G_\mathbb{R}$ acts by conjugation both on $X(\mathbb{C})$ and on the coefficients $(2\pi i)^n\,\mathbb{R}$, which means that the complex $R\Gamma_c(X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})$ carries a natural $G_\mathbb{R}$-action, and we can consider the cohomology of $G_\mathbb{R}$ acting on that complex (this is a particular case of equivariant sheaf cohomology, as introduced by Grothendieck in the Tohoku paper). In terms of cohomology groups, there is a spectral sequence

$$E_2^{pq} = H^p(G_\mathbb{R}, H_c^q(X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})) \Longrightarrow H_c^{p+q}(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}).$$

In this particular case, however, we deal simply with the fixed points of the $G_\mathbb{R}$-action on $H_c^i(X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})$:

$$H_c^i(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) \cong H_c^i(X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})^{G_\mathbb{R}}$$

—this is because for any $\mathbb{Z}/2\mathbb{Z}$-module A, the cohomology groups $H^p(G_\mathbb{R}, A)$ are 2-torsion for $p > 0$, and we deal with $\mathbb{R}$-vector spaces. However, for instance for integral coefficients $(2\pi i)^n\,\mathbb{Z}$, the above spectral sequence does need to degenerate, and the groups $H_c^i(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})$ are more complicated.

3 Regulator

Now to extract the special values of zeta functions, we also need some kind of a regulator (generalizing the regulator of a number field). Normally in this setting, it should be a morphism

$$Reg\colon \begin{array}{c}\text{motivic cohomology}\\ \text{(higher Chow groups)}\end{array} \text{ of } X \longrightarrow \text{Deligne (co)homology of } X(\mathbb{C}).$$

We use the construction of Matt Kerr, James Lewis, and Stefan Müller-Stach from [KLMS2006]. I won't recall the details on Deligne (co)homology here, because it turns out that at the end of the day, thanks to our assumption $n < 0$, things simplify tremendously, and we obtain a morphism

$$Reg\colon R\Gamma(X_{ét}, \mathbb{Z}^c(n)) \to R\Gamma_{BM}(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[1].$$

Unfortunately, the construction of Kerr, Lewis, and Müller-Stach works under a rather severe restriction.

Drawback. *We need to assume that $X_\mathbb{C}$ is smooth and quasi-projective.*

However, since in our particular case of $n < 0$, the regulator has a simpler target ($G_\mathbb{R}$-equivariant Borel–Moore homology of $X(\mathbb{C})$), one might wonder if there is a simpler definition, requiring less from X.

Question. *Is it possible to define a regulator in our setting under less restrictive assumptions on X?*

The regulator is supposed to satisfy the following condition.

Conjecture B(X, n). *The $\mathbb{R}$-dual to the regulator map*

$$Reg^\vee\colon R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[-1] \to R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{R})$$

is a quasi-isomorphism of complexes.

Now let's consider the morphism

$$R\Gamma_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R} \xrightarrow{\smile\theta} R\Gamma_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R}[1]$$

defined by

$$\begin{array}{c} R\Gamma_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R} \\ \big\downarrow\cong \\ R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R})[-1] \;\oplus\; R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})[-1] \\ \big\downarrow \\ R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})[-1] \\ \big\downarrow Reg^\vee \\ R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R}) \\ \big\downarrow \\ R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R}) \;\oplus\; R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R}) \\ \big\downarrow\cong \\ R\Gamma_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R}[1] \end{array}$$

Proposition. *Assume that the conjectures* $\mathbf{L}^c(X_{\acute{e}t},n)$ *and* $\mathbf{B}(X,n)$ *hold. Then the above morphism* $\smile\theta$ *turns* $H^\bullet_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R}$ *into an acyclic complex of finite dimensional vector spaces*

$$\cdots\to H^i_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R} \xrightarrow{\smile\theta} H^{i+1}_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R} \xrightarrow{\smile\theta} H^{i+2}_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R}\to\cdots$$

(this is actually clear from the above definition, once we assume that $Reg^\vee$ is a quasi-isomorphism.)

We now use determinants of complexes, defined by Finn Knudsen and David Mumford in [KM1976]. In the generality we need, the determinant canonically associates to a perfect complex of R-modules a free R-module of rank 1:

$$C^\bullet \rightsquigarrow \det_R C^\bullet.$$

The determinant is functorial on the subcategory given by perfect complexes and *isomorphisms* in $\mathbf{D}(R\text{-}\mathbf{Mod})$, it is compatible with distinguished triangles in a suitable sense, with base change, etc. Without getting into details, let me just say that the properties of the determinant imply that the long exact sequence from the last proposition induces a canonical isomorphism

$$\lambda\colon \mathbb{R} \xrightarrow{\cong} (\det_\mathbb{Z} R\Gamma_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R},$$

allowing us to treat $\det_\mathbb{Z} R\Gamma_{W,c}(X,\mathbb{Z}(n))$ as a lattice in $\mathbb{R}$.

4 The main conjecture about special values

Armed with the morphism λ, we are ready to state our conjecture about vanishing orders and special values of the zeta function.

Conjecture $\mathbf{C}(X,n)$. *For an arithmetic scheme X and $n < 0$*

a) assume that the conjecture $\mathbf{L}^c(X_{ét}, n)$ holds;

b) assume that $X_\mathbb{C}$ is smooth, quasi-projective, so that the regulator morphism Reg exists; assume that the conjecture $\mathbf{B}(X,n)$ holds;

c) assume that $\zeta(X,s)$ has a meromorphic continuation near $s = n$.

Then

1) the special value $\zeta^*(X,n)$ is given up to sign by

$$\lambda(\zeta^*(X,n)^{-1}) \cdot \mathbb{Z} = \det_\mathbb{Z} R\Gamma_{W,c}(X, \mathbb{Z}(n)).$$

2) the vanishing order of $\zeta(X,n)$ at $s = n$ is given by the weighted alternating sum of ranks of the corresponding Weil-étale cohomology groups:

$$\operatorname{ord}_{s=n} \zeta(X,s) = \sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot \operatorname{rk}_\mathbb{Z} H^i_{W,c}(X, \mathbb{Z}(n)).$$

Of course, one can define any kind of complexes and formulate any conjectures about them. The conjecture $\mathbf{C}(X,n)$ is plausible because when X is regular and proper, then it is equivalent to the conjectures stated by Flach and Morin in [FM2016], and they showed in particular that for smooth schemes their special value conjecture is compatible with the **Tamagawa number conjecture** of Bloch, Kato, Fontaine, and Perrin-Riou (see [FPR1994]).

Even for some easy examples, it is not trivial at all to calculate Weil-étale cohomology and verify $\mathbf{C}(X,n)$ directly: among other things, that would require calculation of motivic cohomology. Let us see a couple of examples for the vanishing orders, as it is much easier to count ranks of groups. It is easy to check that under the assumptions a) and b) made in the conjecture, we have

$$\sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot \operatorname{rk}_\mathbb{Z} H^i_{W,c}(X, \mathbb{Z}(n)) = \sum_{i\in\mathbb{Z}} (-1)^i \dim_\mathbb{R} H^i_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R}).$$

Namely, we may use the splitting

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \cong R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \oplus R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R})[-1]$$

and the conjectural quasi-isomorphism

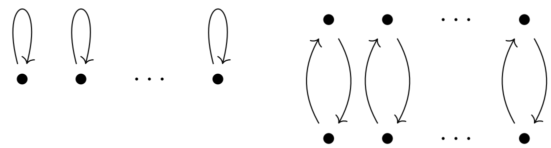
$$Reg^\vee \colon R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R})[-1] \xrightarrow{\cong} R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{R}).$$

This all means that the conjecture actually says that

$$\operatorname{ord}_{s=n} \zeta(X,s) = \chi(R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R}))$$

is the usual Euler characteristic of a very specific and computable complex.

1) For the case of a number ring $X = \mathrm{Spec}\, O_F$, the space $X(\mathbb{C})$ consists of $r_1 + 2\, r_2$ points, corresponding to the real places of F and complex places coming in conjugate pairs:



The complex $R\Gamma_c(X(\mathbb{C}), (2\pi i)^n\, \mathbb{R})$ in this case has just a single $\mathbb{R}$-vector space in degree 0, namely

$$V := ((2\pi i)^n\, \mathbb{R})^{\oplus r_1} \oplus ((2\pi i)^n\, \mathbb{R} \oplus (2\pi i)^n\, \mathbb{R})^{\oplus r_2},$$

where $G_\mathbb{R}$ acts on $((2\pi i)^n\, \mathbb{R})^{\oplus r_1}$ by complex conjugation, while on $((2\pi i)^n\, \mathbb{R} \oplus (2\pi i)^n\, \mathbb{R})^{\oplus r_2}$ the action is given by $(z_1, z_2) \mapsto (\overline{z_2}, \overline{z_1})$ on each summand $(2\pi i)^n\, \mathbb{R} \oplus (2\pi i)^n\, \mathbb{R}$. We see that

$$\chi(R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R})) = \dim_\mathbb{R} V^{G_\mathbb{R}} = \begin{cases} r_2, & n \text{ odd}, \\ r_1 + r_2, & n \text{ even}, \end{cases}$$

and this agrees with vanishing orders of the Dedekind zeta function $\zeta(\mathrm{Spec}\, O_F, s)$ at strictly negative integers.

2) If X is a variety over $\mathbb{F}_q$, then

$$\zeta(X, s) = Z(X, q^{-s}) = \exp\left(\sum_{k \ge 1} \frac{\# X(\mathbb{F}_{q^k})}{k}\, q^{-ks}\right)$$

is Weil's zeta function, which has no zeros or poles for $s < 0$. This is not quite obvious, but it may be seen from the Weil conjectures that if s is a pole or zero, then it should satisfy

$$\mathrm{Re}\, s = i/2, \quad 0 \le i \le 2 \dim X$$

(see e.g. [Kat1994, p. 26–27]). This agrees with the fact that trivially,

$$\chi(R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R})) = 0.$$

5 Stability properties

So we stated the conjecture $\mathbf{C}(X, n)$, and it is equivalent to the conjectures of Flach and Morin for X regular, proper. This is not a big deal, because from the very beginning, the construction of $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ follows theirs, with the difference that in the case $n < 0$ certain things actually become simpler. Now I will explain how new results may be obtained.

The following properties are clear from the definition of the zeta function of an arithmetic scheme:

1) If $U \hookrightarrow X \leftarrow Z$ is an open-closed decomposition, then

$$\zeta(X, s) = \zeta(U, s) \cdot \zeta(Z, s).$$

2) For $r \geq 0$, for the affine space $\mathbb{A}^r_X := \mathbb{A}^r_{\mathbb{Z}} \times X$

$$\zeta(\mathbb{A}^r_X, s) = \zeta(X, s - r).$$

This suggests the following result.

Theorem.

1) *If $U \hookrightarrow X \leftarrow Z$ is an open-closed decomposition of an arithmetic scheme, then if two out of three conjectures $\mathbf{C}(U,n)$, $\mathbf{C}(X,n)$, $\mathbf{C}(Z,n)$ hold, then the other one holds as well.*

2) *The conjecture $\mathbf{C}(\mathbb{A}^r_X, n)$ is equivalent to $\mathbf{C}(X, n-r)$.*

Again, it will be easier to explain this for vanishing orders. As I mentioned, the conjecture actually says that

$$\operatorname{ord}_{s=n} \zeta(X,s) = \chi(R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})).$$

In 1), we have the following distinguished triangle for $G_{\mathbb{R}}$-equivariant cohomology with compact support:

$$R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) \to R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) \to R\Gamma_c(G_{\mathbb{R}}, Z(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) \to \cdots [1]$$

and therefore

$$\chi(R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})) = \chi(R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), (2\pi i)^n\,\mathbb{R})) + \chi(R\Gamma_c(G_{\mathbb{R}}, Z(\mathbb{C}), (2\pi i)^n\,\mathbb{R})).$$

In 2), we need to check that

(*) $$\chi(R\Gamma_c(G_{\mathbb{R}}, \mathbb{C}^r \times X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})) = \chi(R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^{n-r}\,\mathbb{R})),$$

and in fact, we have a $G_{\mathbb{R}}$-equivariant quasi-isomorphism of complexes

$$R\Gamma_c(\mathbb{C}^r \times X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) \simeq R\Gamma_c(X(\mathbb{C}), (2\pi i)^{n-r}\,\mathbb{R})[-2r],$$

where the shift by $-2r$ does not affect the Euler characteristic.

As for the special values part of the conjecture, in 1) one needs to use the "Borel–Moore" triangle

$$R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n)) \to R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)) \to R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)) \to \cdots [1]$$

which in turn gives

$$R\mathrm{Hom}(R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}) \to R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}) \to R\mathrm{Hom}(R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}) \to \cdots [1]$$

This may be combined with the above triangle for $R\Gamma_c(G_{\mathbb{R}}, (-)(\mathbb{C}), (2\pi i)^n\,\mathbb{R})$ and the splittings

$$R\Gamma_{W,c}(-, \mathbb{Z}(n)) \otimes \mathbb{R} \cong R\mathrm{Hom}(R\Gamma((-)_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \oplus R\Gamma_c(G_{\mathbb{R}}, (-)(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[-1]$$

(those are not canonical, but may be chosen in a way compatible with the triangles).

In 2), the key idea is that if $p\colon \mathbb{A}^r_X \to X$ is the canonical projection, then there is a quasi-isomorphism of complexes of sheaves on $X_{\acute{e}t}$

$$Rp_* \mathbb{Z}^c(n) \simeq \mathbb{Z}^c(n-r)[2r],$$

so that

$$R\Gamma(\mathbb{A}^r_{X,\acute{e}t}, \mathbb{Z}^c(n)) \simeq R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n-r))[2r].$$

This corresponds to the formula (*) (the shifts differ by a sign because (*) is written for cohomology with compact support, while the above formula is morally for the dual "Borel–Moore homology").

* * *

All this means that with the established machinery, we can take as a starting point certain schemes for which the conjecture is true (e.g. schemes for which the Tamagawa number conjecture is known), and then, using operations like open-closed decompositions and affine bundles, construct new schemes, possibly singular, for which the conjecture holds as well.

A Definition of Weil-étale complexes

Here I outline the present construction of Weil-étale complexes for the case $n < 0$. Following [FM2016], we consider the complex of torsion sheaves on $X_{ét}$

$$\mathbb{Z}(n) := \text{“}\mathbb{Q}/\mathbb{Z}\text{”}(n)[-1] := \bigoplus_p \varinjlim_r j_{p!}\mu_{p^r}^{\otimes n}[-1].$$

Here $j_p \colon X[1/p] \to X$ is the canonical open immersion, by μ_{p^r} we denote the sheaf of roots of unity on $X[1/p]_{ét}$, and its twist by $n < 0$ is defined by

$$\mu_{p^r}^{\otimes n} := \underline{\mathrm{Hom}}_{X[1/p]}(\mu_{p^r}^{\otimes(-n)}, \mathbb{Z}/p^r).$$

Then the definition of Weil-étale cohomology is summarized by the following diagram in the derived category of abelian groups:

$$\begin{array}{ccccccc}
 & & & & R\Gamma_{W,c}(X,\mathbb{Z}(n)) & & \\
 & & & & \downarrow & & \\
R\mathrm{Hom}(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-2]) & \xrightarrow{\alpha_{X,n}} & R\Gamma_c(X_{ét},\mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X,\mathbb{Z}(n)) & \longrightarrow & R\mathrm{Hom}(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-1]) \\
\downarrow & & \downarrow{\scriptstyle u_\infty^*} & & \vdots{\scriptstyle i_\infty^*} & & \downarrow \\
0 & \longrightarrow & R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}) & \xrightarrow{\mathrm{id}} & R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}) & \longrightarrow & 0[1] \\
 & & & & \downarrow & & \\
 & & & & R\Gamma_{W,c}(X,\mathbb{Z}(n))[1] & &
\end{array}$$

Here is how this diagram is built.

1) Using a duality theorem from [Gei2010], we define a morphism in the derived category of abelian groups
$$\alpha_{X,n} \colon R\mathrm{Hom}(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-2]) \to R\Gamma_c(X_{ét},\mathbb{Z}(n)).$$

2) We pick a cone of $\alpha_{X,n}$ and call it $R\Gamma_{fg}(X,\mathbb{Z}(n))$.

3) Then we define a canonical morphism of complexes
$$u_\infty^* \colon R\Gamma_c(X_{ét},\mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z})$$
(it is a kind of comparison morphism between étale and singular cohomology), and check that
$$u_\infty^* \circ \alpha_{X,n} = 0$$
in the derived category. This implies that there is a morphism *in the derived category*
$$i_\infty^* \colon R\Gamma_{fg}(X,\mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z})$$
(see the diagram).

4) We pick a mapping fiber of i_∞^* and call it $R\Gamma_{W,c}(X,\mathbb{Z}(n))$.

In step 2), it is possible to represent $R\Gamma_{fg}(X, \mathbb{Z}(n))$ by a canonical complex. In step 3), in fact there is a unique i_∞^* sitting in the commutative diagram, but the outlined argument gives it only *as a morphism in the derived category*, not a genuine morphism of complexes. Therefore, $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ is defined only up to a non-canonical isomorphism. This is bad not only for aesthetical reasons, but causes technical problems in practice. This is quite unsatisfactory, but works for our purposes, because $\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n))$ is canonically defined.

Question. *Is there a canonical way to define $R\Gamma_{W,c}(X, \mathbb{Z}(n))$?*

As all the problems come from the non-functoriality of cones, working instead with stable ∞-categories (see [Lur2006]) might be helpful here.

Here are some properties of the complexes and morphisms in the diagram from the previous page:

a) The groups $H^i_c(X_{ét}, \mathbb{Z}(n))$ are $\mathbb{Q}/\mathbb{Z}$-dual to finitely generated abelian groups. In particular, we have $R\Gamma_c(X_{ét}, \mathbb{Z}(n)) \otimes \mathbb{R} \simeq 0$.

b) $H^i_{fg}(X, \mathbb{Z}(n)) := H^i(R\Gamma_{fg}(X, \mathbb{Z}(n)))$ are finitely generated abelian groups (hence the notation "*fg*").

   Moreover, $H^i_{fg}(X, \mathbb{Z}(n)) = 0$ for $i \ll 0$ and it is a finite 2-torsion group for $i \gg 0$. The 2-torsion comes from the real points $X(\mathbb{R})$.

c) The morphism i_∞^* is torsion in the derived category; in particular, $i_\infty^* \otimes \mathbb{R} = 0$.

d) $R\Gamma_c(X(\mathbb{C}), (2\pi i)^n \mathbb{Z})$ is a perfect complex. As for the equivariant cohomology $R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n \mathbb{Z})$, then, unless $X(\mathbb{R}) = \emptyset$, the groups $H^i_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n \mathbb{Z})$ have finite 2-torsion in arbitrarily high degrees, coming from the cohomology of $G_\mathbb{R} \cong \mathbb{Z}/2\mathbb{Z}$. It has the same nature as the 2-torsion in $H^i_{fg}(X, \mathbb{Z}(n))$, and in fact $H^i(i_\infty^*)$ is an isomorphism for $i \gg 0$. As a result, the complex $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ is bounded.

Once we tensor the diagram on the previous page with $\mathbb{R}$, thanks to a) and c), we obtain

$$\begin{array}{ccccc} & & R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} & & \\ & & \downarrow & & \\ R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{R}[-2]) \longrightarrow 0 & \longrightarrow & R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes \mathbb{R} & \xrightarrow{\cong} & R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{R}[-1]) \\ & & \downarrow{\scriptstyle 0} & & \\ & & R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n \mathbb{R}) & & \\ & & \downarrow & & \\ & & R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R}[1] & & \end{array}$$

this explains the splitting

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \cong R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \oplus R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n \mathbb{R})[-1].$$

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