Stellingen

behorende bij het proefschrift

Zeta-values of arithmetic schemes at negative integers and Weil-étale cohomology

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In everything what follows, X is an arithmetic scheme (separated, of finite type over $\operatorname{Spec}\mathbb{Z}$) and n is a *strictly negative* integer.

We denote by $\mathbb{Z}^c(n)$ the dualizing Bloch's cycle complex of sheaves on $X_{ét}$, and by $\mathbb{Z}(n)$ the complex of sheaves $\bigoplus_p \varinjlim_r j_{p!}\mu_{p^r}^{\otimes n}[-1]$, where $j_p\colon X[1/p] \hookrightarrow X$ is the canonical open immersion for each prime p and $\mu_{p^r}^{\otimes n}$ is the sheaf of p^r-th roots of unity on $X[1/p]_{ét}$ twisted by n.

We denote by $R\Gamma_c(X_{ét},\mathcal{F}^\bullet)$ the étale cohomology with compact support and by $R\widehat{\Gamma}_c(X_{ét},\mathcal{F}^\bullet)$ the modified étale cohomology with compact support, as defined e.g. in Milne's book "Arithmetic Duality theorems".

For brevity, we write $[A^\bullet, B^\bullet]$ instead of $R\mathrm{Hom}(A^\bullet, B^\bullet)$.

All the main constructions are done assuming the **conjecture** $\mathbf{L}^c(X_{ét},n)$: *the cohomology groups $H^i(X_{ét},\mathbb{Z}^c(n))$ are finitely generated for all $i\in\mathbb{Z}$.*

I. Assuming $\mathbf{L}^c(X_{ét},n)$, there is a quasi-isomorphism of complexes

$$R\widehat{\Gamma}_c(X_{ét},\mathbb{Z}(n)) \xrightarrow{\cong} [R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}/\mathbb{Z}[-2]].$$

II. Assume $\mathbf{L}^c(X_{ét},n)$ and let $\alpha_{X,n}$ be the composition of morphisms of complexes

$$[R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-2]] \to [R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}/\mathbb{Z}[-2]] \xleftarrow{\cong} R\widehat{\Gamma}_c(X_{ét},\mathbb{Z}(n)) \to R\Gamma_c(X_{ét},\mathbb{Z}(n))$$

Let $R\Gamma_{fg}(X,\mathbb{Z}(n))$ be a cone of $\alpha_{X,n}$:

$$[R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-2]] \xrightarrow{\alpha_{X,n}} R\Gamma_c(X_{ét},\mathbb{Z}(n)) \to R\Gamma_{fg}(X,\mathbb{Z}(n)) \to [R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-1]]$$

Then the cohomology groups $H^i(R\Gamma_{fg}(X,\mathbb{Z}(n)))$ are finitely generated, trivial for $i \ll 0$, and only have 2-torsion for $i \gg 0$.

III. For any prime ℓ the group $H^i_c(X_{\overline{\mathbb{Q}},ét},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_\mathbb{Q}}$ has no nontrivial divisible elements.

IV. Assume $\mathbf{L}^c(X_{ét},n)$ and let $\alpha_{X,n}$ be as above. Let

$$u_\infty^*\colon R\Gamma_c(X_{ét},\mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z})$$

be the canonical comparison morphism, discussed in §§0.7–0.8 of the thesis. Then $u_\infty^*\circ\alpha_{X,n} = 0$. Let i_∞^* be a morphism of complexes defined via

$$\begin{array}{ccccccc}
[R\Gamma(X,\mathbb{Z}^c(n)),\mathbb{Q}[-2]] & \xrightarrow{\alpha_{X,n}} & R\Gamma_c(X_{ét},\mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X,\mathbb{Z}(n)) & \longrightarrow & \cdots \\
\downarrow & & \downarrow{\scriptstyle u_\infty^*} & & \downarrow{\scriptstyle i_\infty^*} & & \downarrow \\
0 & \longrightarrow & R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}) & \xrightarrow{\mathrm{id}} & R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

and let $R\Gamma_{W,c}(X,\mathbb{Z}(n))$ be a mapping fiber of i_∞^*:

$$R\Gamma_{W,c}(X,\mathbb{Z}(n)) \to R\Gamma_{fg}(X,\mathbb{Z}(n)) \xrightarrow{i_\infty^*} R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}) \to R\Gamma_{W,c}(X,\mathbb{Z}(n))[1]$$

Then $R\Gamma_{W,c}(X,\mathbb{Z}(n))$ is a perfect complex.

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To formulate the next result, we denote by $\mathbf{C}(X,n)$ the following conjecture.

a) *assume that the conjecture $\mathbf{L}^c(X_{ét},n)$ holds;*

b) *assume that $X_{\mathbb{C}}$ is smooth, quasi-projective, so that the regulator morphism*

$$Reg\colon R\Gamma(X_{ét},\mathbb{Z}^c(n)) \to R\Gamma_{BM}(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})[1]$$

exists and assume the ***regulator conjecture****: the $\mathbb{R}$-dual is a quasi-isomorphism*

$$Reg^\vee\colon R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})[-1] \xrightarrow{\cong} [R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{R}].$$

c) *assume that the zeta-function $\zeta(X,s)$ has a meromorphic continuation near $s=n$.*

Then

1) *the leading coefficient $\zeta^*(X,n)$ of the Taylor expansion of $\zeta(X,s)$ at $s=n$ is given up to sign by*

$$\lambda(\zeta^*(X,n)^{-1})\cdot\mathbb{Z} = \det\nolimits_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)),$$

where λ is the trivialization morphism defined using the regulator in §2.3 of the thesis;

2) *the vanishing order of $\zeta(X,n)$ at $s=n$ is given by the weighted alternating sum of ranks of $H^i_{W,c}(X,\mathbb{Z}(n))$:*

$$\operatorname{ord}_{s=n}\zeta(X,s) = \sum_{i\in\mathbb{Z}}(-1)^i\cdot i\cdot \operatorname{rk}_{\mathbb{Z}} H^i_{W,c}(X,\mathbb{Z}(n)).$$

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V. The conjecture $\mathbf{C}(X,n)$ is compatible with disjoint unions, open-closed decompositions and taking affine bundles in the following sense.

- If $X=\coprod_{0\le i\le r}X_i$ is a disjoint union of arithmetic schemes, then the conjectures $\mathbf{C}(X_i,n)$ for $i=0,\dots,r$ together imply $\mathbf{C}(X,n)$.
- If $U\hookrightarrow X \hookleftarrow Z$ is an open-closed decomposition of an arithmetic scheme, then if two out of three conjectures $\mathbf{C}(U,n)$, $\mathbf{C}(Z,n)$, $\mathbf{C}(X,n)$ hold, the other one holds as well.
- The conjecture $\mathbf{C}(\mathbb{A}^r_X,n)$ is equivalent to $\mathbf{C}(X,n-r)$.

VI. Sometimes it is possible to talk about unique cones in the derived category. For a distinguished triangle $A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \xrightarrow{w} A^\bullet[1]$ assume that $A^\bullet$ is a complex such that $H^i(A^\bullet)$ are finite dimensional $\mathbb{Q}$-vector spaces and $C^\bullet$ is "almost perfect", meaning that $H^i(C^\bullet)$ are finitely generated groups, zero for $i\ll 0$ and have only 2-torsion for $i\gg 0$. Then the cone of u is unique up to a unique isomorphism in the derived category.

VII. If A and B are finitely generated abelian groups, then up to equivalence, every extension of $\operatorname{Hom}(B,\mathbb{Q}/\mathbb{Z})$ by $\operatorname{Hom}(A,\mathbb{Q}/\mathbb{Z})$ is $\mathbb{Q}/\mathbb{Z}$-dual to an extension of A by B.

VIII. The order of zero of the Dedekind zeta function of a number field K at $n<0$ may be interpreted via the equivariant cohomology of $X=\operatorname{Spec}O_K$ as $\operatorname{ord}_{s=n}\zeta_K(s)=\dim_{\mathbb{R}}H^0_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})$.